SECTION-02 - BE-02

APTITUDE TEST (MATHEMATICS & SOFT SKILL)

MATHEMATICS

1. Determinant and Matrices
2. Trigonometry
3. Vectors
4. Co-ordinate Geometry
5. Function & Limit
6. Differentiation and its Applications
7. Integration
8. Logarithm
9. Statistics

ENGLISH

10. Comprehension of Unseen Passage
11. Theory of Communication
12. Grammar
13. Correction of Incorrect Words and Sentences

4. Co-ordinate Geometry

[1] Introduction :

In the previous standards we studied pure geometry. This geometry was Euclidean geometry. In which the principles of geometry and diagrams were used. Algebra has provided important backing to the development and logical study of geometry. The French mathematician Rene Descartes first time in the history of mathematics, combined algebra and Euclidean geometry. This geometry is therefore called Cartesian Geometry after Descartes. He used coordinates to denote the point. Hence this geometry is also known as Coordinate Geometry.

[2] Revision :

In standard 10, we have learned the following results.

(1) **Distance formula in R^2 :** If $A(x_1, y_1), B(x_2, y_2) \in R^2$ then the distance between A and B can be obtained using the following formula.

$$AB = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

This formula is known as Distance formula in R^2.

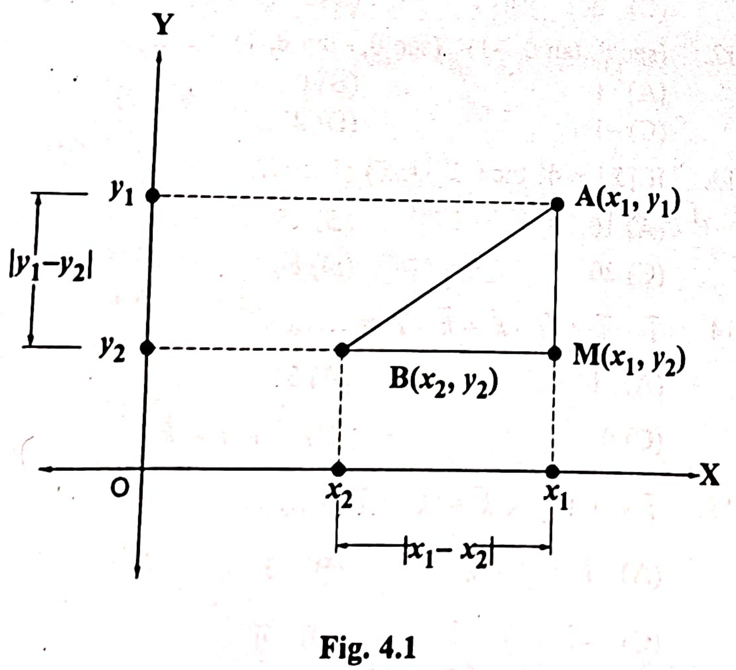


Fig. 4.1

(2) **Section formula :** The coordinates of the point P dividing the line segment joining $A(x_1, y_1)$ and $B(x_2, y_2)$ in the ratio $m : n$ are $\left(\dfrac{mx_2 + nx_1}{m+n}, \dfrac{my_2 + ny_1}{m+n}\right)$.

(3) **Mid point of a line segment :** If (x, y) is the coordinate of the mid point M of the line segment AB joining $A(x_1, y_1)$ and $B(x_2, y_2)$, then

$$(x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

(4) **Area of a triangle :** The area of the triangle ABC with the vertices

$A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ can be obtained by using the following formula.

$$\text{Area of } \Delta\, ABC = \frac{1}{2}|D|,$$

$$\text{where } D = x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)$$

$$= \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

[A] Line

[3] Line :

At the school level we learned that two different points on a plane determine a unique line. Thus a line is a set of points determined by two points. In Standard 9 and 10 we studied linear equations of two variables and their graphs. If $a, b, c \in R$ and a and b are not zero simultaneously, then $ax + by + c = 0$ is called a linear equation of two variables and its graph is a line. In this chapter we will study about equations of line, their different forms · · · etc.

[4] Cartesian equation of a line in R^2 :

(1) **Equation of a line perpendicular to Y-axis :**

Let $A(x_1, b)$ and $B(x_2, b)$ be two distinct points on $\overleftrightarrow{AB}$, where b is a constant real number. It can be seen from the Fig. 4.2 that the y-coordinate of every point on $\overleftrightarrow{AB}$ is b.

$\therefore$ The equation of $\overleftrightarrow{AB}$ is represented as $y = b$.

X-axis is also perpendicular to Y-axis and y-coordinate of every point on X-axis is zero. So the equation of X-axis is $y = 0$.

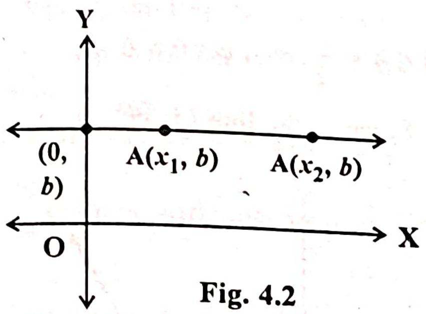


Fig. 4.2

Thus the equation of X-axis or a line perpendicular to Y-axis i.e. parallel to X-axis is of the form $y = b$, where $b \in R$.

(2) Equation of a line perpendicular to X-axis :

Let A(a, y_1) and B(a, y_2) be two distinct points on $\overleftrightarrow{AB}$, where a is a constant real number. It can be seen from the Figure 4.3 that the x-coordinate of every point on $\overleftrightarrow{AB}$ is a.

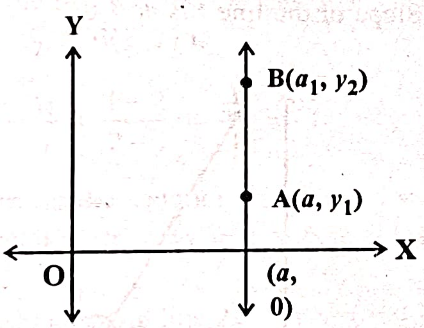


Fig. 4.3

$\therefore$ The equation of $\overleftrightarrow{AB}$ is represented as $x = a$.

Y-axis is also perpendicular to X-axis and x-coordinate of every point on Y-axis is zero. So the equation of Y-axis is $x = 0$.

Thus the equation of Y-axis or a line perpendicular to X-axis i.e. parallel to Y-axis is of the form $x = a$, where $a \in R$.

(3) Equation of a line not perpendicular to the axis :

Let A(x_1, y_1) and B(x_2, y_2) be two distinct points in R^2, then the equation of $\overleftrightarrow{AB}$ is $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$

Also $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$ is satisfied by taking $x_1 = x_2 = a$ or $y_1 = y_2 = b$ for the equations $x = a$ and $y = b$ of lines perpendicular to the axis respectively. Thus the equation of any line AB is of the form $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$.

[5] Condition for three points to be collinear :

Suppose A(x_1, y_1), B(x_2, y_2) and C(x_3, y_3) are collinear points. Now as shown in the section 4.4(3), equation of $\overleftrightarrow{BC}$ is $\begin{vmatrix} x & y & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$.

Since A, B and C are collinear $\therefore$ A $\in \overleftrightarrow{BC}$.

So A(x_1, y_1) satisfies the equation of $\overleftrightarrow{BC}$.

$$\therefore \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Thus the necessary and sufficient condition for three distinct points A(x_1, y_1), B(x_2, y_2) and C(x_3, y_3) to be collinear is $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

Example-1 : Find cartesian equations of lines passing through the following pairs of points.

(i) (2, 3), (3, –1)

Solution :

(i) Let A(x_1, y_1) = A(1, 2) and B(x_2, y_2) = B(3, –2).

$$\therefore \text{ Equation of } \overleftrightarrow{AB} : \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

$$\therefore \begin{vmatrix} x & y & 1 \\ 2 & 3 & 1 \\ 3 & -1 & 1 \end{vmatrix} = 0$$

$\therefore$ $x(3+1) - y(2-3) + 1(-2-9) = 0$

$\therefore$ $4x + y - 11 = 0$

$\therefore$ The required equation is $4x + y - 11 = 0$

[6] Linear equation in R^2 :

The equation $ax + by + c = 0$, $a, b, c \in R$, $a^2 + b^2 \neq 0$ is called linear equation in R^2. In the Example 1, we have seen that the equations of lines are linear equations of the form $ax + by + c = 0$, $a^2 + b^2 \neq 0$ in R^2.

Thus, the graph of a linear equation in R^2 is a line and equation of any line is a linear equation in R^2.

[7] Slope of a line :

If $A(x_1, y_1)$ and $B(x_2, y_2)$ are distinct points and the line joining them is not perpendicular to X-axis, then we define the slope of $\overleftrightarrow{AB}$ by $\dfrac{y_2 - y_1}{x_2 - x_1}$ and we denote it by m.

Thus the slope of a non vertical line passing through $A(x_1, y_1)$ and $B(x_2, y_2)$ is $\boxed{m = \dfrac{y_2 - y_1}{x_2 - x_1}}$ ... (i)

Now if the line l, which is not perpendicular to the axis, intersects the X-axis at P and $\angle APX = \theta$ then it is said that the line makes an angle of measure θ with the positive direction of X-axis.

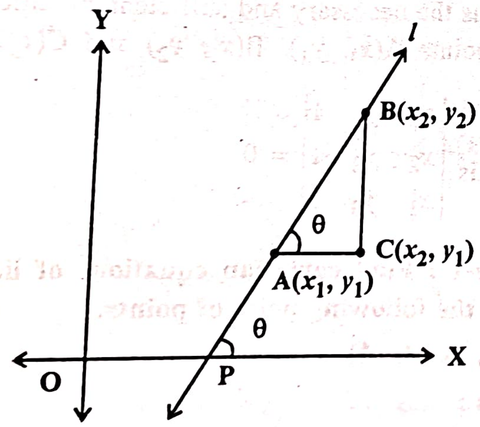


Fig. 4.4

Now from the Figure 4.4, $m\angle BAC = m\angle APX = \theta$

$\therefore$ In the right angle triangle ABC

$$\tan\theta = \frac{BC}{AC} = \frac{y_2 - y_1}{x_2 - x_1} = \text{Slope of } \overleftrightarrow{AB} = m$$

If a non vertical line makes an angle of measure θ $(0 < \theta < \pi, \theta \neq \frac{\pi}{2})$ with the positive direction of X-axis, then the slope of that line is $m = \tan\theta$ . ... (ii)

If $0 < \theta < \frac{\pi}{2}$ then $\tan\theta > 0$

$\therefore$ Slope of the line l is $m > 0$.

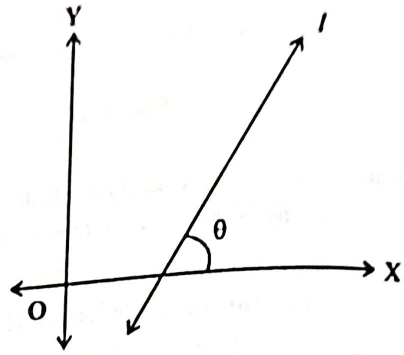


Fig. 4.5

If $\frac{\pi}{2} < \theta < \pi$ then $\tan\theta < 0$.

$\therefore$ Slope of the line l is $m < 0$.

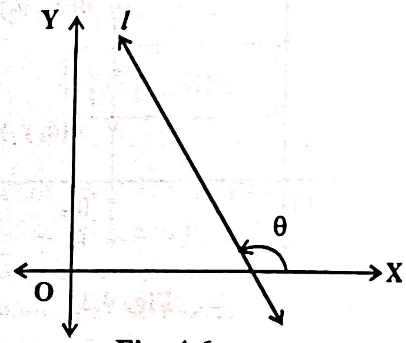


Fig. 4.6

If the line is perpendicular to Y-axis, then the slope of that line is $m = 0$.

If the line is perpendicular to X-axis, then its slope is undefined.

Also if the Cartesian equation of a line is $ax + by + c = 0$, $a^2 + b^2 \neq 0$, then the slope of the line represented by this equation is

$$m = -\frac{a}{b} \text{ ; if } b \neq 0 \quad \text{... (iii)}$$

$= \text{undefined; if } b = 0$

Example-2 : Do as directed :

(i) Find the slope of the line passing through (8, 5) and (1, –2).

(ii) Find the slope of the line passing through (1, 2) and (2, 1).

(iii) **Find the slope of the line $2x - 3y + 5 = 0$.**

(iv) **Find the slope of the line $(\cos \alpha)\, x + (\sin \alpha)\, y = 5$.**

(v) **Find the slope of the line making an angle of measure $\frac{\pi}{4}$ radian with the positive direction of X-axis.**

(vi) **Find the measure of an angle made by the line $x - \sqrt{3}y + 1 = 0$ with the positive direction of X-axis.**

Solution :

(i) Slope of the line passing through $A(x_1, y_1) = A(8, 5)$ and $B(x_2, y_2) = B(1, -2)$ is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2 - 5}{1 - 8} = \frac{-7}{-7} = 1$$

(ii) Slope of the line passing through $A(x_1, y_1) = A(1, 2)$ and $B(x_2, y_2) = B(2, 1)$ is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1 - 2}{2 - 1} = \frac{-1}{1} = -1$$

(iii) By comparing the equation $2x - 3y + 5 = 0$ with the linear equation $ax + by + c = 0$ in R^2, $a = 2$ and $b = -3$.

$\therefore$ The slope of the line $2x - 3y + 5 = 0$ is

$$m = -\frac{a}{b} = -\frac{2}{(-3)} = \frac{2}{3}$$

(iv) By comparing the equation $(\cos \alpha)\, x + (\sin \alpha)\, y = 5$ with the linear equation $ax + by + c = 0$ in R^2, $a = \cos \alpha$ and $b = \sin \alpha$.

$\therefore$ The slope of the line $(\cos \alpha)\, x + (\sin \alpha)\, y = 5$ is

$$m = -\frac{a}{b} = -\frac{\cos \alpha}{\sin \alpha} = -\cot \alpha$$

(v) The measure of the angle made by the line with the positive direction of X-axis is $\theta = \frac{\pi}{4}$ radian.

$\therefore$ The slope of the line is $m = \tan \theta = \tan \frac{\pi}{4} = 1$

(vi) Slope of the line $x - \sqrt{3}y + 1 = 0$ is

$$m = -\frac{a}{b} = -\frac{1}{(-\sqrt{3})} = \frac{1}{\sqrt{3}}$$

Suppose this line makes an angle of measure θ with the positive direction of X-axis.

$$\therefore \quad \tan \theta = m = \frac{1}{\sqrt{3}}$$

$$\therefore \quad \theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

[8] Necessary and sufficient condition for two distinct lines to be parallel :

If two distinct lines are perpendicular to X-axis, then their slopes are undefined and if the slopes of two distinct lines are undefined, then they are perpendicular to X-axis. Thus they are parallel. (See the figure 4.7)

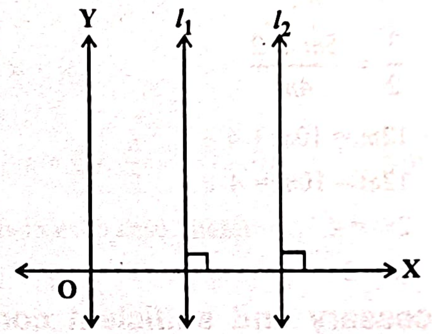


Fig. 4.7

If none of the distinct lines l_1 and l_2 is perpendicular to x-axis, θ_1 and θ_2 are the measures of angles made by them with the positive direction of X-axis and m_1 and m_2 are their slopes respectively, then

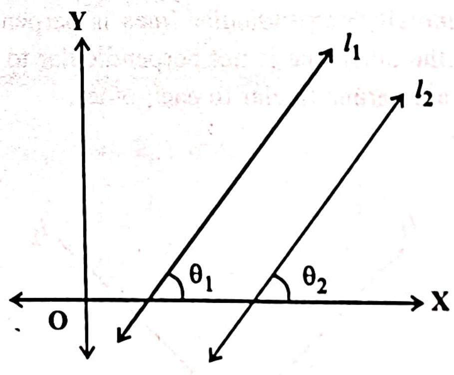


Fig. 4.8

$m_1 = \tan \theta_1$ and $m_2 = \tan \theta_2$

Now, $l_1 \parallel l_2 \Leftrightarrow \theta_1 = \theta_2$ (From the figure 4.8)

$\Leftrightarrow \tan \theta_1 = \tan \theta_2$

$\Leftrightarrow m_1 = m_2$

Thus two given distinct lines are parallel if and only if their slopes are equal or undefined.

Example-3 : Do as directed :

If two lines $3mx - 2my - 10 = 0$ and $(5m + 2)x - 4my - 28 = 0$ are parallel to each other, then find the value of m.

Solution :

Slope of the line $3mx - 2my - 10 = 0$ is

$$m_1 = -\frac{3m}{(-2m)} = \frac{3}{2}$$

and slope of the line $(5m + 2)x - 4my - 28 = 0$ is

$$m_2 = -\frac{(5m+2)}{(-4m)} = \frac{5m+2}{4m}$$

Now, the given lines are parallel.

$\therefore \quad m_1 = m_2$

$\therefore \quad \frac{3}{2} = \frac{5m+2}{4m}$

$\therefore \quad 12m = 10m + 4$

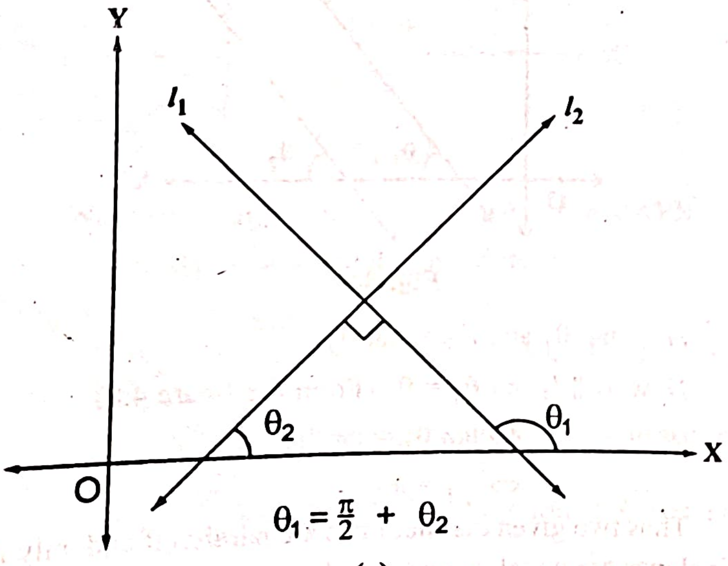
$\therefore \quad 12m - 10m = 4$

$\therefore \quad 2m = 4$

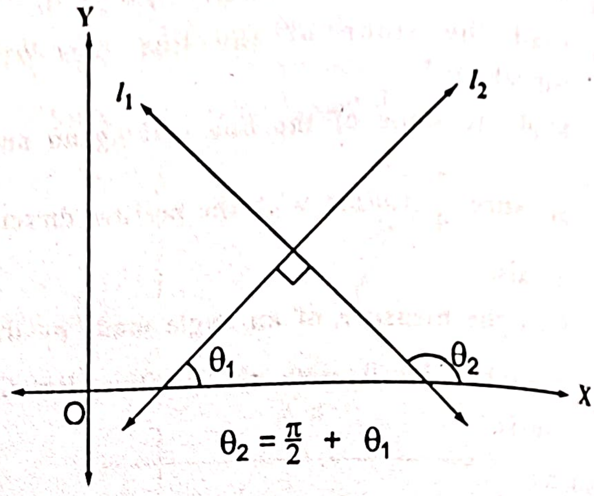
$\therefore \quad m = 2$

[9] Necessary and sufficient condition for two distinct lines to be perpendicular to each other :

If one line is perpendicular to X-axis and the other line is perpendicular to Y -axis then both the lines are perpendicular to each other while it is not possible that one of the mutually perpendicular lines is perpendicular to X-axis and the other line is not perpendicular to Y-axis, since the axes are perpendicular to each other.



(a)



(b)

Fig. 4.9

Suppose none of the two distinct lines l_1 and l_2 perpendicular to any axis. Let the slopes of the lines l_1 and be m_1 and m_2 and their measures of angles be θ_1 and respectively.

$\therefore m_1 = \tan\theta_1$ and $m_2 = \tan\theta_2$

It is clear that $m_1 \neq 0$, $m_2 \neq 0$ and $0 < \theta_1 < \theta_2 <$

$\theta_1 \neq \frac{\pi}{2}$, $\theta_2 \neq \frac{\pi}{2}$.

Now, if $l_1 \perp l_2$, then it is clear from the Figure 4.9 t

$$\theta_1 = \frac{\pi}{2} + \theta_2 \text{ or } \theta_2 = \frac{\pi}{2} + \theta_1$$

$\therefore \quad l_1 \perp l_2 \Leftrightarrow \theta_1 = \frac{\pi}{2} + \theta_2$ or $\theta_2 = \frac{\pi}{2} + \theta_1$

$\Leftrightarrow \tan\theta_1 = \tan\left(\frac{\pi}{2} + \theta_2\right)$ or $\tan\theta_2 = \tan\left(\frac{\pi}{2} + \theta\right.$

$\Leftrightarrow \tan\theta_1 = -\cot\theta_2$ or $\tan\theta_2 = -\cot\theta_1$

$\Leftrightarrow \tan\theta_1 = \frac{-1}{\tan\theta_2}$ or $\tan\theta_2 = -\frac{1}{\tan\theta_1}$

$\Leftrightarrow \tan\theta_1 \cdot \tan\theta_2 = -1$

$\Leftrightarrow m_1 \cdot m_2 = -1$

Thus if none of the two lines is perpendicular to axis, then necessary and sufficient condition for two dist lines to be perpendicular to each other is $m_1 \cdot m_2 = -1$, wh m_1 and m_2 are the slopes of these two lines.

Example-4 : Do as directed :

(i) **Prove that the lines $7x + y - 1 = 0$ and $3x - 21y + 2 = 0$ are perpendicular to each other.**

Solution :

(i) Slope of the line $7x + y - 1 = 0$ is

$$m_1 = -\frac{a}{b} = -\frac{7}{1} = -7$$

and slope of the line $3x - 21y + 2 = 0$ is

$$m_2 = -\frac{a}{b} = -\frac{3}{(-21)} = \frac{1}{7}$$

Now, $m_1 \cdot m_2 = (-7) \cdot \left(\frac{1}{7}\right) = -1$

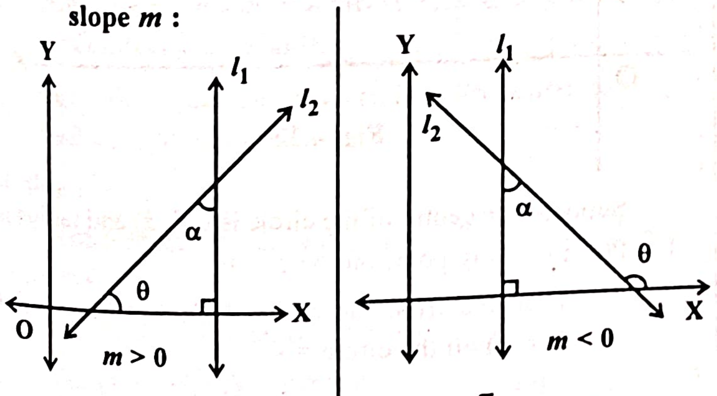
$\therefore$ The given lines are perpendicular to each other.

[10] Angle between two intersecting lines :

If two distinct lines in R^2 are not parallel to each other, then they intersect each other at a unique point.

If two lines are perpendicular to each other, then the measure of the angle between them is $\frac{\pi}{2}$ radian. If two lines are not perpendicular to each other, then the radian measure of the acute angle out of two pairs of congruent vertically opposite angles formed at their point of intersection is called the measure of the angle between two lines. If we denote the measure of this angle by α, then $0 < \alpha < \frac{\pi}{2}$.

(1) **Measure of the angle between two lines, when one line is perpendicular to X-axis and the other has slope m :**



$\theta + \alpha = \frac{\pi}{2}$ (a) $\quad$ $\frac{\pi}{2} + \alpha = \theta$ (b)

Fig. 4.10

Suppose l_1 is perpendicular to X-axis and slope of the line l_2 is m ($m \neq 0$). If line l_2 makes an angle of measure θ with the positive direction of X-axis, then $m = \tan \theta$. Let the angle between these two lines is α ($0 < \alpha < \frac{\pi}{2}$).

From the Figure 4.10 (a), if $m > 0$ i.e. $0 < \theta < \frac{\pi}{2}$, then $\theta + \alpha = \frac{\pi}{2}$.

$$\therefore \quad \alpha = \frac{\pi}{2} - \theta = \left|\frac{\pi}{2} - \theta\right| \qquad (\because 0 < \theta < \frac{\pi}{2})$$

and from the Figure 4.10 (b), if $m < 0$ i.e. $\frac{\pi}{2} < \theta < \pi$, then $\frac{\pi}{2} + \alpha = \theta$.

$$\therefore \quad \alpha = \theta - \frac{\pi}{2} = \left|\frac{\pi}{2} - \theta\right| \quad (\because \frac{\pi}{2} < \theta < \pi)$$

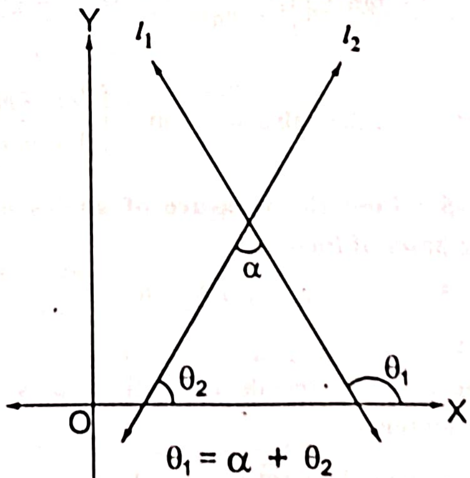
Thus in each case, measure of the angle between two lines is $\alpha = \left|\frac{\pi}{2} - \theta\right|$.

(2) **Measure of the angle between two lines, neither of them is perpendicular to X-axis :**

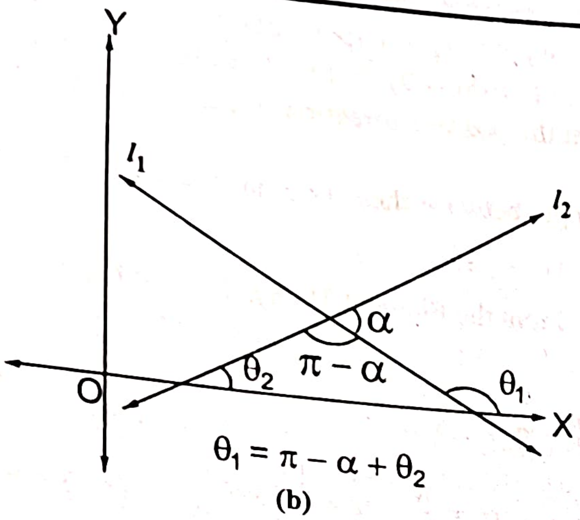
Suppose none of the lines l_1 and l_2 is perpendicular to X-axis. Let slopes of the lines l_1 and l_2 be m_1 and m_2 and measure of their angles with the positive direction of X-axis be θ_1 and θ_2 respectively,

where $0 \le \theta_1, \theta_2 < \frac{\pi}{2}$ and $\theta_1 \neq \frac{\pi}{2}$, $\theta_2 \neq \frac{\pi}{2}$

$\therefore \quad m_1 = \tan \theta_1$ and $m_2 = \tan \theta_2$



(a)



(b)

Fig. 4.11

Suppose $\theta_1 > \theta_2$

$\therefore$ It can be seen from the figure 4.11 (a) and 4.11 (b),

$\theta_1 = \alpha + \theta_2$ or $\theta_1 = \pi - \alpha + \theta_2$

$\therefore$ $\alpha = \theta_1 - \theta_2$ or $\alpha = \pi - (\theta_1 - \theta_2)$

$\therefore$ $\tan \alpha = \tan (\theta_1 - \theta_2)$ or $\tan \alpha = \tan (\pi - (\theta_1 - \theta_2))$

$= -\tan (\theta_1 - \theta_2)$

$\therefore$ $\tan \alpha = |\tan (\theta_1 - \theta_2)|$

$$\left(\because 0 \le \alpha < \frac{\pi}{2} \therefore \tan\alpha \ge 0\right)$$

$$= \left|\frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \cdot \tan\theta_2}\right|$$

$$= \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$\therefore \quad \alpha = \tan^{-1}\left(\left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|\right)$$

If $\theta_2 > \theta_1$, then also $\alpha = \tan^{-1}\left(\left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|\right)$ can be

Example-5 : Find the measure of angles between the following pairs of lines :

$x = 5$, $\quad \sqrt{3}x - y + 7 = 0$

Solution :

Line $x = 5$ is perpendicular to the X-axis and its slope is undefined.

Slope of the line $\sqrt{3}x - y + 7 = 0$ is

$$m = -\frac{\sqrt{3}}{(-1)} = \sqrt{3}$$

Let the angle of the line $\sqrt{3}x - y + 7 = 0$ with positive direction of X-axis be θ then $\tan\theta = m = \sqrt{3}$.

$$\therefore \quad \theta = \frac{\pi}{3}$$

$\therefore$ If the angle between the given two lines is α, th

$$\alpha = \left|\frac{\pi}{2} - \theta\right| = \left|\frac{\pi}{2} - \frac{\pi}{3}\right| = \left|\frac{3\pi - 2\pi}{6}\right| = \frac{\pi}{6}.$$

[B] Circle

[11] Circle :

Set of all the points at a constant distance from a fixe point in a plane is called circle. The constant distance called the radius of the circle and the fixed point is called th centre of the circle.

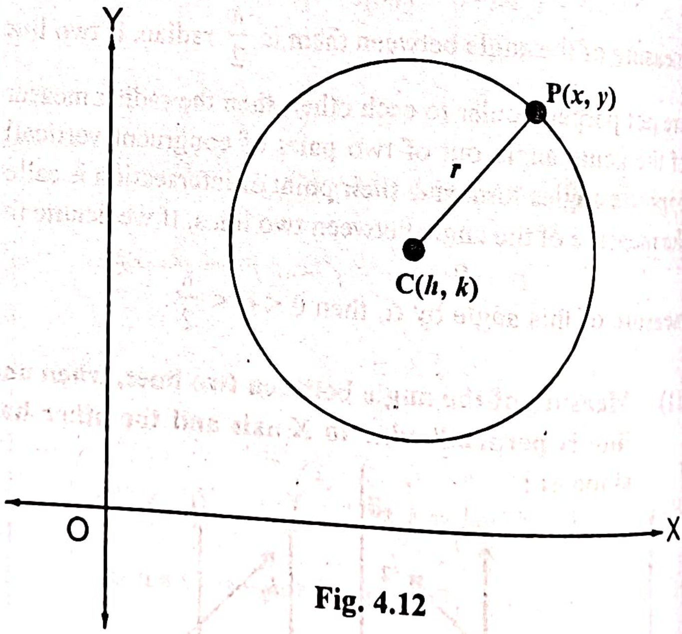


Fig. 4.12

Suppose the centre of the circle is $C(h, k)$ and radius r. Let $P(x, y)$ be any point on the circle.

$\therefore$ Distance from the centre $C(h, k)$ to any poir $P(x, y)$ on the circle $= r$

$\therefore$ $CP = r$

$\therefore$ $CP^2 = r^2$

$\therefore$ $(x - h)^2 + (y - k)^2 = r^2$

Thus the cartesian equation of a circle with centr $C(h, k)$ and radius r is $(x - h)^2 + (y - k)^2 = r^2$ ... (i)

[12] Standard form of the equation of a circle :

An equation of a circle whose centre is origin and radius is r is called standard form of the equation of circle.

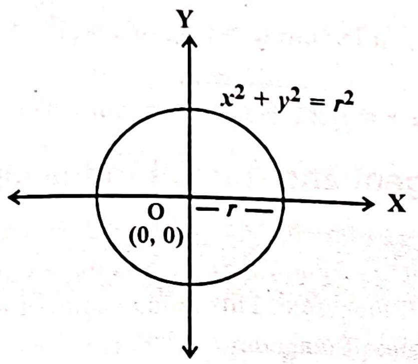


Fig. 4.13

$\therefore$ Taking $h = k = 0$ in the equation (i),

$(x - 0)^2 + (y - 0)^2 = r^2$

$\therefore$ $x^2 + y^2 = r^2$ ... (ii)

Equation (ii) is called the standard form of the equation of a circle.

Further, in the equation (ii), if radius $r = 1$ the standard form reduces to $x^2 + y^2 = 1$. This is called the equation of the unit circle.

Example-6 : Do as directed :

(i) **Find the equation of a circle with centre (–2, 5) and radius 4.**

(ii) **Find the equation of a circle with centre (2, –3) and radius 3.**

(iii) **Find the equation of a circle with centre (a cos α, a sin α) and radius a.**

(iv) **Find the equation of a circle with centre (4, 5) and circumference 8π unit.**

(v) **Find the equation of a circle with centre (–3, –2) and area 9π sq. unit.**

Solution :

(i) Centre of the circle is $(h, k) = (-2, 5)$ and radius is $r = 4$.

$\therefore$ Required equation of circle is

$(x - h)^2 + (y - k)^2 = r^2$

$\therefore$ $(x - (-2))^2 + (y - 5)^2 = (4)^2$

$\therefore$ $(x + 2)^2 + (y - 5)^2 = 16$

$\therefore$ $x^2 + 4x + 4 + y^2 - 10y + 25 - 16 = 0$

$\therefore$ $x^2 + y^2 + 4x - 10y + 13 = 0$

(ii) Centre of the circle is $(h, k) = (2, -3)$ and radius is $r = 3$.

$\therefore$ Required equation of circle is

$(x - h)^2 + (y - k)^2 = r^2$

$\therefore$ $(x - 2)^2 + (y - (-3))^2 = (3)^2$

$\therefore$ $(x - 2)^2 + (y + 3)^2 = 9$

$\therefore$ $x^2 - 4x + 4 + y^2 + 6y + 9 - 9 = 0$

$\therefore$ $x^2 + y^2 - 4x + 6y + 4 = 0$

(iii) Centre of the circle is $(h, k) = (a \cos\alpha, \ a \sin\alpha)$ and radius is $r = a$.

$\therefore$ Required equation of circle is

$(x - h)^2 + (y - k)^2 = r^2$

$\therefore$ $(x - a\cos\alpha)^2 + (y - a\sin\alpha)^2 = a^2$

$\therefore$ $x^2 - 2ax\cos\alpha + a^2\cos^2\alpha + y^2 - 2ay\sin\alpha + a^2\sin^2\alpha = a^2$

$\therefore$ $x^2 + y^2 - 2a\,x\cos\alpha - 2a\,y\sin\alpha + a^2(\cos^2\alpha + \sin^2\alpha) - a^2 = 0$

$\therefore$ $x^2 + y^2 - 2ax\cos\alpha - 2ay\sin\alpha + a^2 - a^2 = 0$ $(\because \cos^2\alpha + \sin^2\alpha = 1)$

$\therefore$ $x^2 + y^2 - 2ax\cos\alpha - 2ay\sin\alpha = 0$

(iv) Centre of the circle is $(h, k) = (4, 5)$ and radius is r.

Now, circumference of the circle = 8π

$\therefore$ $2\pi r = 8\pi$ $\quad\therefore r = \dfrac{8\pi}{2\pi} = 4$

$\therefore$ Required equation of circle is

$(x - h)^2 + (y - k)^2 = r^2$

$\therefore$ $(x - 4)^2 + (y - 5)^2 = (4)^2$

$\therefore$ $x^2 - 8x + 16 + y^2 - 10y + 25 = 16$

$\therefore$ $x^2 + y^2 - 8x - 10y + 25 = 0$

(v) Centre of the circle is $(h, k) = (-3, -2)$ and suppose the radius is r.

Now, area of the circle = 9π

$\therefore$ $\pi r^2 = 9\pi$

$\therefore$ $r^2 = 9$

$\therefore$ $r = 3$ $\quad(\because r > 0)$

$\therefore$ Required equation of circle is

$(x - h)^2 + (y - k)^2 = r^2$

$\therefore$ $(x - (-3))^2 + (y - (-2))^2 = (3)^2$

$\therefore$ $(x + 3)^2 + (y + 2)^2 = 9$

$\therefore$ $x^2 + 6x + 9 + y^2 + 4y + 4 = 9$

$\therefore$ $x^2 + y^2 + 6x + 4y + 4 = 0$

[13] General form of the equation of a Circle :

As discussed before, the equation of a circle with centre (h, k) and radius r is

$$(x-h)^2 + (y-k)^2 = r^2.$$

$\therefore \quad x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - r^2 = 0$

It is clear from this equation that

(i) The equation of any circle is a quadratic equation in two variables.

(ii) In the equation of a circle, coefficients of x^2 and y^2 are non-zero and equal.

(iii) The coefficient of xy-term in the equation is zero.

(**Note :** (ii) and (iii) are the necessary conditions for the quadratic equation in R^2 to represent a circle.)

Thus we take the general form of the equation of a circle in the form $x^2 + y^2 + 2gx + 2fy + c = 0$

Now let us determine the condition for the equation $x^2 + y^2 + 2gx + 2fy + c = 0$ to represent a circle, its centre and radius.

Here $x^2 + y^2 + 2gx + 2fy + c = 0$

$\therefore \quad x^2 + 2gx + g^2 + y^2 + 2fy + f^2 = g^2 + f^2 - c$

$\therefore \quad (x+g)^2 + (y+f)^2 = g^2 + f^2 - c$

$\therefore \quad$ If $g^2 + f^2 - c > 0$, then

$$(x+g)^2 + (y+f)^2 = \left(\sqrt{g^2 + f^2 - c}\right)^2$$

The above equation shows that the distance of the moving point (x, y) from the point $(-g, -f)$ is $\sqrt{g^2 + f^2 - c}$.

Thus the equation $x^2 + y^2 + 2gx + 2fy + c = 0$ represent a circle if $g^2 + f^2 - c > 0$ and in this case its centre is $C(-g, -f)$ and radius is $r = \sqrt{g^2 + f^2 - c}$.

Example-7 : If the following equations represent a circle then find the centre and radius :

$$x^2 + y^2 - 2x + 4y + 1 = 0$$

Solution :

In the given equation $x^2 + y^2 - 2x + 4y + 1 = 0$, the coefficient of x^2 and y^2 are equal and the coefficient of xy is zero.

Comparing the given equation with the general form $x^2 + y^2 + 2gx + 2fy + c = 0$,

$2g = -2, \quad 2f = 4, \quad c = 1$

$\therefore \quad g = -1, \quad f = 2$

Now, $g^2 + f^2 - c = (-1)^2 + (2)^2 - 1$

$= 1 + 4 - 1 = 4 > 0$

$\therefore$ The given equation represents a circle

Centre of the circle $= (-g, -f) = (1, -2)$ and

radius $r = \sqrt{g^2 + f^2 - c} = \sqrt{4} = 2$

[14] Tangent and normal to the circle :

Tangent of a circle : A line lying in the plane of the circle and intersecting the circle at a unique point is called the tangent of the circle. This unique point of intersection is called the point of tangency.

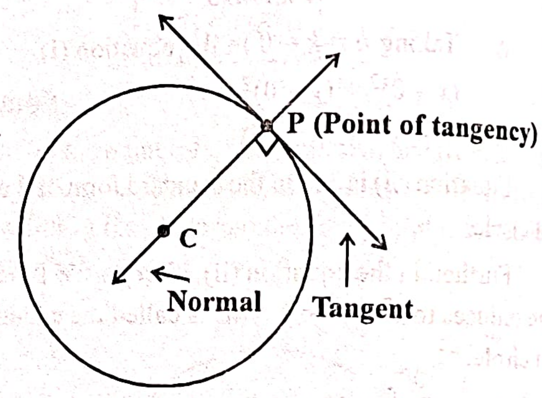


Fig. 4.14

Normal of a circle : A line lying in the plane of the circle and perpendicular to the tangent of the circle at the point of tangency is called the normal of the circle.

Normal of the circle always passes through the centre of the circle.

(1) Tangent and normal at a point (x_1, y_1) on the circle $x^2 + y^2 = r^2$:

At a point (x_1, y_1) on the circle $x^2 + y^2 = r^2$, the equation of the tangent is $x_1x + y_1y = r^2$ and the equation of the normal is $y_1x - x_1y = 0$.

(2) Tangent and normal at a point (x_1, y_1) on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$:

At a point (x_1, y_1) on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, the equation of the tangent is $x_1x + y_1y + g(x + x_1) + f(y + y_1) + c = 0$ and the equation of the normal is $\dfrac{x - x_1}{x_1 + g} = \dfrac{y - y_1}{y_1 + f}$

Example-8 : Find the equation of tangent and normal to the following circle at the given point on it :

Equation of circle	Point on the circle
$x^2 + y^2 = 52$	$(-6, 4)$

Solution :

A point on the circle $x^2 + y^2 = 52$ is $(x_1, y_1) = (-6, 4)$

Equation of tangent : $x_1x + y_1y = 52$

$\therefore \quad -6x + 4y = 52$

$\therefore \quad 3x - 2y = -26$

$\therefore \quad 3x - 2y + 26 = 0$

Equation of tangent : $y_1x - x_1y = 0$

$\therefore \quad 4x - (-6)y = 0$

$\therefore \quad 4x + 6y = 0$

$\therefore \quad 2x + 3y = 0$

Multiple Choice Questions (MCQs) (Solution with Explanation)

1. Slope of the line passing through the points (8, 5) and (1, –2) is ....

(A) 1 (B) –1 (C) –7 (D) 7

Ans. : (A)

Explanation : Slope of the line passing through $A(x_1, y_1) = A(8, 5)$ and $B(x_2, y_2) = B(1, -2)$ is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2-5}{1-8} = \frac{-7}{-7} = 1$$

2. Slope of the line passing through the points (1, 2) and (2, 1) is ....

(A) $\frac{1}{2}$ (B) 2 (C) 1 (D) –1

Ans. : (D)

Explanation : Slope of the line passing through $A(x_1, y_1) = A(1, 2)$ and $B(x_2, y_2) = B(2, 1)$ is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1-2}{2-1} = \frac{-1}{1} = -1$$

3. If slope of the line passing through the points (x, 6) and (–2, 5) is $\frac{1}{3}$, then $x =$ ....

(A) –1 (B) 1 (C) 5 (D) 3

Ans. : (B)

Explanation : Slope of the line passing through (x, 6) and $(-2, 5) = \frac{1}{3}$

$\therefore \quad \frac{6-5}{x-(-2)} = \frac{1}{3} \qquad \therefore \quad \frac{1}{x+2} = \frac{1}{3}$

$\therefore \quad x + 2 = 3 \qquad \therefore \quad x = 1$

4. Slope of the line making an angle of measure $\frac{\pi}{4}$ with the positive direction of X-axis is ....

(A) 1 (B) $\frac{1}{\sqrt{3}}$

(C) $\sqrt{3}$ (D) ∞

Ans. : (A)

Explanation : Measure of the angle made by the line with the positive direction of X-axis is $\theta = \frac{\pi}{4}$ radian.

$\therefore$ Slope of the line is $m = \tan\theta = \tan\frac{\pi}{4} = 1$

5. If the slope of the line is $\sqrt{3}$, then its angle with the positive direction of X-axis is ....

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{6}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{2}$

Ans. : (C)

Explanation : Suppose the angle of the line with the positive direction of X-axis is θ.

$\therefore \quad \tan\theta$ = slope of the line $= \sqrt{3}$

$\therefore \quad \theta = \tan^{-1}\left(\sqrt{3}\right) = \frac{\pi}{3}$

6. Slope of the line $2x - 3y + 5 = 0$ is ....

(A) $-\frac{2}{3}$ (B) $-\frac{3}{2}$

(C) $\frac{2}{3}$ (D) $\frac{3}{2}$

Ans. : (C)

Explanation : By comparing the equation $2x - 3y + 5 = 0$ with the linear equation $ax + by + c = 0$ in R^2, $a = 2$ and $b = -3$.

$\therefore$ Slope of the line $2x - 3y + 5 = 0$ is

$$m = -\frac{a}{b} = -\frac{2}{(-3)} = \frac{2}{3}$$

7. Slope of the line $(\cos\alpha)x + (\sin\alpha)y = 5$ is ....

(A) $\tan\alpha$ (B) $\cot\alpha$

(C) $-\tan\alpha$ (D) $-\cot\alpha$

Ans. : (D)

Explanation : By comparing the equation $(\cos\alpha)x + (\sin\alpha)y - 5 = 0$ with the linear equation $ax + by + c = 0$ in R^2 $a = \cos\alpha$ and $b = \sin\alpha$.

$\therefore$ Slope of the line $(\cos\alpha)x + (\sin\alpha)y - 5 = 0$ is

$$m = -\frac{a}{b} = -\frac{\cos\alpha}{\sin\alpha} = -\cot\alpha$$

8. Angle between the line $x + y = 0$ and $x - y = 0$ is .....

(A) $\frac{\pi}{3}$ (B) $\frac{\pi}{6}$ (C) 0 (D) $\frac{\pi}{2}$

Ans. : (D)

Explanation :

Slope of the line $x + y = 1$ is $m_1 = -\frac{a}{b} = -\frac{1}{1} = -1$ and

Slope of the line $x - y = 1$ is $m_2 = -\frac{a}{b} = -\frac{1}{(-1)} = 1$

Now, $m_1 \cdot m_2 = (-1)\cdot(1) = -1$

$\therefore$ Both the lines are perpendicular to each other.

Hence the angle between them is $\frac{\pi}{2}$.

9. X-intercept of the line $3x + 5 = 0$ is ....

(A) $-\frac{5}{3}$ (B) $\frac{5}{3}$

(C) $\frac{3}{5}$ (D) 0

Ans. : (A)

Explanation : For the line $3x + 5 = 0$, $a = 3$, $b = 0$, $c = 5$

$\therefore$ X-intercept $= -\frac{c}{a} = -\frac{5}{3}$

10. X-intercept of the line $x\cos\alpha + y\sin\alpha = 2$ is ....

(A) $2\cos\alpha$ (B) $2\sec\alpha$

(C) $-2\cos\alpha$ (D) $-2\sec\alpha$

Ans. : (B)

Explanation : For the line $x\cos\alpha + y\sin\alpha - 2 = 0$, $a = \cos\alpha$, $b = \sin\alpha$, $c = -2$

$\therefore$ X-intercept $= -\frac{c}{a} = -\frac{(-2)}{\cos\alpha} = 2\sec\alpha$

11. If the equation $x^2 + (p+5)xy + y^2 + 12x - 18y - 5 = 0$ is the equation of a circle, then $p =$ ....

(A) 0 (B) 5 (C) -5 (D) $\frac{1}{5}$

Ans. : (C)

Explanation : As the equation $x^2 + (p+5)xy + y^2 + 12x - 18y - 5 = 0$ represent a circle, the coefficient of xy must be zero.

$\therefore$ $p + 5 = 0$ $\therefore$ $p = -5$

12. Centre of the circle $x^2 + y^2 + 3x - 4y - 4 = 0$ is ......

(A) $\left(-\frac{3}{2}, 2\right)$ (B) $\left(\frac{3}{2}, -2\right)$

(C) $\left(-\frac{3}{2}, -2\right)$ (D) $\left(\frac{3}{2}, 2\right)$

Ans. : (A)

Explanation : By comparing the equation $x^2 + y^2 + 3x - 4y - 4 = 0$ of the circle, with the general form $x^2 + y^2 + 2gx + 2fy + c = 0$,

$2g = 3$, $2f = -4$

$\therefore$ $g = \frac{3}{2}$, $f = -2$

$\therefore$ Centre of the circle $= (-g, -f) = \left(-\frac{3}{2}, 2\right)$

13. If the centre of the circle $x^2 + y^2 + 3px - 1 = 0$ is $(6, 0)$, then $p =$ ....

(A) 4 (B) -4 (C) 2 (D) -2

Ans. : (B)

Explanation : By comparing the equation $x^2 + y^2 + 3px - 1 = 0$ of the circle, with the general form $x^2 + y^2 + 2gx + 2fy + c = 0$,

$2g = 3p$, $2f = 0$

$\therefore\ g = \frac{3p}{2},\ f = 0$

Now, centre of the circle = (6, 0)

$\therefore\ (-g, -f) = (6, 0)$

$\therefore\ -g = 6 \quad \therefore\ -\frac{3p}{2} = 6 \therefore\ p = -4$

14. Radius of the circle $x^2 + y^2 = 18$ is ....

(A) $2\sqrt{3}$ (B) $3\sqrt{2}$ (C) $2\sqrt{2}$ (D) $3\sqrt{3}$

Ans. : (B)

Explanation : By comparing the equation $x^2 + y^2 = 18$ of the circle, with the standard form $x^2 + y^2 = r^2$, $r^2 = 18$.

$\therefore\ r = \sqrt{18} = 3\sqrt{2}$

15. Radius of the circle $x^2 + y^2 + gx - fy = 0$ is ....

(A) $4\sqrt{g^2 + f^2}$ (B) $2\sqrt{g^2 + f^2}$

(C) $\frac{1}{2}\sqrt{g^2 + f^2}$ (D) $\frac{1}{4}\sqrt{g^2 + f^2}$

Ans. : (C)

Explanation : By comparing the equation $x^2 + y^2 + gx - fy = 0$ of the circle, with the general form $x^2 + y^2 + 2Gx + 2Fy + C = 0$,

$2G = g,\quad 2F = -f,\quad C = 0$

$\therefore\ G = \frac{g}{2}\quad F = -\frac{f}{2}$

$\therefore$ Radius of the circle $r = \sqrt{G^2 + F^2 - C}$

$= \sqrt{\left(\frac{g}{2}\right)^2 + \left(-\frac{f}{2}\right)^2 - 0}$

$= \sqrt{\frac{g^2}{4} + \frac{f^2}{4}}$

$= \sqrt{\frac{g^2 + f^2}{4}} = \frac{1}{2}\sqrt{g^2 + f^2}$

16. The equation of the tangent to the circle $x^2 + y^2 = 25$ at a point $(0, -5)$ is ....

(A) $y + 5 = 0$ (B) $y - 5 = 0$

(C) $x + 5 = 0$ (D) $x - 5 = 0$

Ans. : (A)

Explanation : The equation of the tangent to the circle $x^2 + y^2 = 25$ at a point $(x_1, y_1) = (0, -5)$ is

$x_1x + y_1y = 25$

$\therefore\ (0)\,x + (-5)\,y = 25$

$\therefore\ -5y = 25$

$\therefore\ y = -5$

$\therefore\ y + 5 = 0$

17. The equation of the normal to the circle $x^2 + y^2 = 10$ at a point $(\sqrt{10}, 0)$ is ....

(A) $y = \sqrt{10}$ (B) $x = \sqrt{10}$

(C) $y = 0$ (D) $y = -\sqrt{10}$

Ans. : (C)

Explanation : The equation of the normal to the circle $x^2 + y^2 = 10$ at a point $(x_1, y_1) = (\sqrt{10}, 0)$ is :

$y_1x - x_1y = 0$

$\therefore\ (0)x - \sqrt{10}y = 0$

$\therefore\ -\sqrt{10}y = 0$

$\therefore\ y = 0$

Multiple Choice Questions (MCQ's) with (Final Answers)

1. The equation of a line passing through (5, 0) and perpendicular to the X-axis is ....

(A) $y = 0$ (B) $x = 5$

(C) $x + y = 5$ (D) $x - y = 5$

2. The equation of a line passing through (0, –4) and perpendicular to the Y-axis is ....

(A) $y + 4 = 0$ (B) $y - 4 = 0$

(C) $x = 0$ (D) $x - y = 4$

3. The equation of a line passing through (3, 0) and (0, 4) is ....

(A) $x + y = 7$ (B) $4x - 3y = 0$

(C) $4x + 3y = 12$ (D) $3x + 4y = 25$

4. Slope of the line $x + 5 = 0$ is ....

(A) –5 (B) 0

(C) $\frac{1}{5}$ (D) undefined

5. Slope of the line $y - 3 = 0$ is ....

(A) 3 (B) 0 (C) –3 (D) $\frac{1}{3}$

6. Slope of the line $2x - 5y + 3 = 0$ is ....

(A) $\frac{2}{5}$ (B) $-\frac{2}{5}$ (C) $\frac{5}{2}$ (D) $-\frac{5}{2}$

7. Slope of the line $3x - 2y + 8 = 0$ is ....

(A) $\frac{3}{2}$ (B) $\frac{2}{3}$ (C) $-\frac{3}{2}$ (D) $-\frac{2}{3}$

8. Slope of the line $5x - y + 3 = 0$ is ....

(A) –5 (B) 5 (C) $\frac{1}{5}$ (D) $-\frac{1}{5}$

9. Slope of the line $2x + y - 8 = 0$ is ....

(A) $-\frac{1}{2}$ (B) $\frac{1}{2}$ (C) 2 (D) –2

10. Slope of the line $4y - 2x + 1 = 0$ is ....

(A) $-\frac{1}{2}$ (B) $\frac{1}{2}$

(C) 2 (D) –2

11. Slope of the line perpendicular to the line $5x - 7y + 3 = 0$ is ....

(A) $\frac{5}{7}$ (B) $\frac{7}{5}$ (C) $-\frac{7}{5}$ (D) $\frac{7}{3}$

12. Slope of the line perpendicular to the line $2x + 3y = 7$ is ....

(A) $\frac{2}{3}$ (B) $\frac{3}{2}$ (C) $-\frac{2}{3}$ (D) $-\frac{3}{2}$

13. If two lines having slopes m_1 and m_2 are parallel to each other then ....

(A) $m_1 m_2 = -1$ (B) $m_1 m_2 = 1$

(C) $m_1 m_2 = 0$ (D) $m_1 = m_2$

14. If two lines having slopes m_1 and m_2 are parallel to each other then ....

(A) $m_1 m_2 = 1$ (B) $m_1 m_2 = -1$

(C) $m_1 m_2 = 0$ (D) $m_1 = m_2$

15. Angle of the line $x - 5 = 0$ with the positive direction of X-axis is ....

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{3}$

(C) $\frac{\pi}{2}$ (D) 0

16. The equation of the line parallel to the line $x + 2y + 2 = 0$ is ....

(A) $2x + y + 2 = 0$ (B) $2x - y + 2 = 0$

(C) $x + 2y + 3 = 0$ (D) $x - 2y + 2 = 0$

17. The equation of the line parallel to the line $3x - 2y + 12 = 0$ is ....

(A) $3x - 2y - 12 = 0$ (B) $2x - 3y + 12 = 0$

(C) $3x + 2y + 12 = 0$ (D) $2x + 3y + 12 = 0$

18. X-intercept of the line $2x + 3y - 4 = 0$ is ....

(A) 2 (B) –2

(C) $\frac{1}{2}$ (D) $-\frac{1}{2}$

19. If the Y-intercept of the line $3x - 5y + k = 0$ is 2, then $k =$ ....

(A) 6 (B) –6 (C) 10 (D) –10

20. Y-intercept of line $x + 2 = 0$ is ....

(A) –2 (B) 2

(C) 0 (D) does not exist

21. Centre of the circle $x^2 + y^2 - 2x + 4y + 1 = 0$ is ....

(A) (1, 2) (B) (1, –2)

(C) (–1, –2) (D) (–1, 2)

22. Centre of the circle $x^2 + y^2 - 4x + 7 = 0$ is ....

(A) (–2, 0) (B) (2, 0) (C) (– 4, 0) (D) (0, 4)

23. Radius of the circle $x^2 + y^2 - 2x + 4y + 1 = 0$ is ......

(A) $\sqrt{6}$ (B) 2 (C) 4 (D) 6

24. Radius of the circle $x^2 + y^2 = 25$ is ....

(A) –5 (B) 25 (C) 5 (D) ± 5

25. Radius of the circle $x^2 + y^2 = 50$ is ....

(A) $5\sqrt{2}$ (B) $2\sqrt{5}$

(C) 5 (D) 10

26. Radius of the circle $x^2 + y^2 - 4x - 6y + 4 = 0$ is ....

(A) 4 (B) 6 (C) 2 (D) 3

27. Radius of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is ....

(A) $g^2 + f^2 - c$ (B) $\sqrt{g^2 + f^2 - c}$

(C) $\sqrt{g^2 + f^2 + c}$ (D) $\sqrt{g^2 + f^2 - c^2}$

28. The equation of the tangent to the circle $x^2 + y^2 = 13$ at a point (2, 3) on it is ....

(A) $2x - 3y = 13$ (B) $3x - 2y = 13$

(C) $2x + 3y = 0$ (D) $2x + 3y = 13$

29. The equation of the tangent to the circle $x^2 + y^2 = 25$ at a point (4, 3) on it is ....

(A) $4x + 3y = 0$ (B) $3x + 4y = 0$

(C) $3x - 4y = 0$ (D) $4x - 3y = 0$

Answers

(1) B	(2) A	(3) C	(4) D	(5) B
(6) A	(7) A	(8) B	(9) D	(10) B
(11) C	(12) B	(13) D	(14) B	(15) C
(16) C	(17) D	(18) A	(19) C	(20) D
(21) C	(22) A	(23) B	(24) C	(25) A
(26) D	(27) B	(28) D	(29) C	

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