Term Work Assignment- 1 Solution

Subject Name & Code:

Mathematics- I - BE01R00041

TWA-1: Indeterminate Forms and L'Hospital's Rule

(**Disclaimer:** The purpose of these AI-generated responses is just education and reference. Utilise them to grasp topics and structure, but always rewrite in your own words and double-check the content before submitting. Academic misuse is not the creator's fault.)

1. $\lim_{x\to 0} \frac{e^x+e^{-x}-x^2-2}{\sin^2 x - x^2}$

Answer:

Step 1: Check Indeterminate Form

Substitute $x = 0$:

Numerator: $e^0 + e^0 - 0 - 2 = 1 + 1 - 2 = 0$

Denominator: $\sin^2(0) - 0 = 0$

This is of the form $\frac{0}{0}$. Apply L'Hospital's Rule.

Step 2: Apply L'Hospital's Rule (Differentiate Numerator and Denominator)

$$\text{Let } L = \lim_{x\to 0} \frac{\frac{d}{dx}[e^x + e^{-x} - x^2 - 2]}{\frac{d}{dx}[\sin^2 x - x^2]} = \lim_{x\to 0} \frac{e^x - e^{-x} - 2x}{2\sin x \cos x - 2x}$$

Simplify denominator: $2\sin x \cos x = \sin 2x$

$$L = \lim_{x\to 0} \frac{e^x - e^{-x} - 2x}{\sin 2x - 2x}$$

Step 3: Check Form Again ($\frac{0}{0}$) and Apply L'Hospital's Rule Again

$$L = \lim_{x\to 0} \frac{\frac{d}{dx}[e^x - e^{-x} - 2x]}{\frac{d}{dx}[\sin 2x - 2x]} = \lim_{x\to 0} \frac{e^x + e^{-x} - 2}{2\cos 2x - 2}$$

Step 4: Substitute $x = 0$

Numerator: $e^0 + e^0 - 2 = 1 + 1 - 2 = 0$

Denominator: $2\cos 0 - 2 = 2(1) - 2 = 0$

Still $\frac{0}{0}$. Apply L'Hospital's Rule a third time.

Step 5: Apply L'Hospital's Rule Third Time

$$L = \lim_{x \to 0} \frac{\frac{d}{dx}[e^x + e^{-x} - 2]}{\frac{d}{dx}[2\cos 2x - 2]} = \lim_{x \to 0} \frac{e^x - e^{-x}}{-4\sin 2x}$$

Step 6: Substitute $x = 0$

Numerator: $e^0 - e^0 = 1 - 1 = 0$

Denominator: $-4\sin 0 = 0$

Still $\frac{0}{0}$. Apply L'Hospital's Rule a fourth time.

Step 7: Apply L'Hospital's Rule Fourth Time

$$L = \lim_{x \to 0} \frac{\frac{d}{dx}[e^x - e^{-x}]}{\frac{d}{dx}[-4\sin 2x]} = \lim_{x \to 0} \frac{e^x + e^{-x}}{-8\cos 2x}$$

Step 8: Final Evaluation

Substitute $x = 0$:

$$L = \frac{e^0 + e^0}{-8\cos 0} = \frac{1+1}{-8(1)} = \frac{2}{-8} = -\frac{1}{4}$$

Final Answer:

$$\boxed{-\frac{1}{4}}$$

2. $\lim_{x \to 1} \frac{x^x - x}{x - 1 - \log x}$

Answer:

Step 1: Check Indeterminate Form

Substitute $x = 1$:

Numerator: $1^1 - 1 = 0$

Denominator: $1 - 1 - \log(1) = 0 - 0 = 0$

This is $\frac{0}{0}$. Apply L'Hospital's Rule.

Step 2: Differentiate Numerator and Denominator

Let $f(x) = x^x - x$. To differentiate x^x, let $y = x^x$. Then $\ln y = x \ln x$.

$$\frac{1}{y}\frac{dy}{dx} = \ln x + 1 \implies \frac{dy}{dx} = x^x(\ln x + 1)$$

Therefore, the derivative of the numerator is:

$$f'(x) = x^x(\ln x + 1) - 1$$

The derivative of the denominator $g(x) = x - 1 - \log x$ is:

$$g'(x) = 1 - \frac{1}{x}$$

Step 3: Apply L'Hospital's Rule Once

$$L = \lim_{x\to 1}\frac{f'(x)}{g'(x)} = \lim_{x\to 1}\frac{x^x(\ln x + 1) - 1}{1 - \frac{1}{x}} = \lim_{x\to 1}\frac{x^x(\ln x + 1) - 1}{\frac{x-1}{x}} = \lim_{x\to 1}\frac{x\left[x^x(\ln x + 1) - 1\right]}{x - 1}$$

Step 4: Check Form Again ($\frac{0}{0}$)

Substitute $x = 1$:

Numerator: $1 \cdot [1^1(\ln 1 + 1) - 1] = 1 \cdot [1(0 + 1) - 1] = 1 \cdot (1 - 1) = 0$

Denominator: $1 - 1 = 0$

Apply L'Hospital's Rule again to this new expression.

Let $h(x) = x\left[x^x(\ln x + 1) - 1\right]$ and $k(x) = x - 1$. We need $\frac{dh}{dx}$.

Step 5: Differentiate $h(x)$ (Requires Product Rule and Chain Rule)

$$h(x) = x \cdot A(x) - x, \quad \text{where } A(x) = x^x(\ln x + 1)$$

First, find $A'(x)$. Let $A(x) = B(x) \cdot C(x)$, where $B(x) = x^x$ and $C(x) = \ln x + 1$.
We know $B'(x) = x^x(\ln x + 1)$ from before.
$C'(x) = \frac{1}{x}$
Using the product rule:

$$A'(x) = B'(x)C(x) + B(x)C'(x) = [x^x(\ln x + 1)](\ln x + 1) + x^x \cdot \frac{1}{x} = x^x(\ln x + 1)^2 + \frac{x^x}{x}$$

Now, apply the product rule to $h(x) = x \cdot A(x) - x$:

$$h'(x) = \frac{d}{dx}[x \cdot A(x)] - \frac{d}{dx}[x] = [1 \cdot A(x) + x \cdot A'(x)] - 1 = A(x) + xA'(x) - 1$$

Substitute $A(x)$ and $A'(x)$:

$$h'(x) = x^x(\ln x + 1) + x\left[x^x(\ln x + 1)^2 + \frac{x^x}{x}\right] - 1 = x^x(\ln x + 1) + x^{x+1}(\ln x + 1)^2 + x^x - 1$$

The derivative of the denominator $k(x) = x - 1$ is $k'(x) = 1$.

Step 6: Apply L'Hospital's Rule Again

$$L = \lim_{x\to 1} \frac{h'(x)}{k'(x)} = \lim_{x\to 1} \left[x^x(\ln x + 1) + x^{x+1}(\ln x + 1)^2 + x^x - 1\right]$$

Step 7: Evaluate the Limit by Substituting $x = 1$

$$x^x|_{x=1} = 1, \quad \ln(1) = 0$$

So,

$$L = [1 \cdot (0+1)] + [1^2 \cdot (0+1)^2] + [1] - 1 = (1) + (1 \cdot 1) + 1 - 1 = 1 + 1 + 1 - 1 = 2$$

Final Answer for Question 2:

$$\boxed{2}$$

3. $\lim_{x\to 0} \frac{\tan x - \sin x}{\sin^3 x}$

Answer:

Step 1: Check Indeterminate Form

Substitute $x = 0$: $\tan 0 - \sin 0 = 0$, $\sin^3 0 = 0$. $\frac{0}{0}$.

Step 2: Simplify Algebraically Before Applying L'Hospital's

$$\frac{\tan x - \sin x}{\sin^3 x} = \frac{\frac{\sin x}{\cos x} - \sin x}{\sin^3 x} = \frac{\sin x(1/\cos x - 1)}{\sin^3 x} = \frac{1/\cos x - 1}{\sin^2 x} = \frac{1 - \cos x}{\cos x \cdot \sin^2 x}$$

Now, $1 - \cos x = 2\sin^2(x/2)$ and $\sin^2 x = 4\sin^2(x/2)\cos^2(x/2)$. Substitute:

$$= \frac{2\sin^2(x/2)}{\cos x \cdot 4\sin^2(x/2)\cos^2(x/2)} = \frac{2}{4\cos x \cos^2(x/2)} = \frac{1}{2\cos x \cos^2(x/2)}$$

Step 3: Evaluate the Limit

$$\lim_{x\to 0} \frac{1}{2\cos x \cos^2(x/2)} = \frac{1}{2 \cdot \cos 0 \cdot \cos^2(0)} = \frac{1}{2 \cdot 1 \cdot 1} = \frac{1}{2}$$

Final Answer:

$$\boxed{\frac{1}{2}}$$

4. $\lim_{x \to 0} \frac{\cosh x - \cos x}{x \sin x}$

Answer:

Step 1: Check Indeterminate Form

Substitute $x = 0$:

Numerator: $\cosh(0) - \cos(0) = 1 - 1 = 0$

Denominator: $0 \cdot \sin(0) = 0 \cdot 0 = 0$

$\frac{0}{0}$. Apply L'Hospital's Rule.

Step 2: Apply L'Hospital's Rule (Differentiate Numerator and Denominator)

$$\frac{d}{dx}[\cosh x - \cos x] = \sinh x + \sin x$$

$$\frac{d}{dx}[x \sin x] = \sin x + x \cos x \quad \text{(using product rule)}$$

So,

$$L = \lim_{x \to 0} \frac{\sinh x + \sin x}{\sin x + x \cos x}$$

Step 3: Check Form Again ($\frac{0}{0}$)

Substitute $x = 0$: Numerator: $0 + 0 = 0$, Denominator: $0 + 0 \cdot 1 = 0$.

Apply L'Hospital's Rule again.

Step 4: Apply L'Hospital's Rule Again

Differentiate numerator: $\frac{d}{dx}[\sinh x + \sin x] = \cosh x + \cos x$

Differentiate denominator: $\frac{d}{dx}[\sin x + x \cos x] = \cos x + [1 \cdot \cos x + x \cdot (-\sin x)] = \cos x + \cos x - x \sin x = 2 \cos x - x \sin x$

So,

$$L = \lim_{x \to 0} \frac{\cosh x + \cos x}{2 \cos x - x \sin x}$$

Step 5: Evaluate the Limit

Substitute $x = 0$:

$$L = \frac{\cosh 0 + \cos 0}{2\cos 0 - 0} = \frac{1+1}{2 \cdot 1} = \frac{2}{2} = 1$$

Final Answer:

$$\boxed{1}$$

5. $\lim_{x \to a} \frac{x^a - a^x}{x^x - a^a}$

Answer:

Step 1: Check Indeterminate Form

Substitute $x = a$:

Numerator: $a^a - a^a = 0$

Denominator: $a^a - a^a = 0$

$\frac{0}{0}$. Apply L'Hospital's Rule.

Step 2: Differentiate Numerator and Denominator

Let $f(x) = x^a - a^x$. Then,

$$f'(x) = ax^{a-1} - a^x \ln a$$

Let $g(x) = x^x - a^a$. To differentiate x^x, as before, $\frac{d}{dx}[x^x] = x^x(\ln x + 1)$. So,

$$g'(x) = x^x(\ln x + 1) - 0 = x^x(\ln x + 1)$$

Step 3: Apply L'Hospital's Rule

$$L = \lim_{x \to a} \frac{ax^{a-1} - a^x \ln a}{x^x(\ln x + 1)}$$

Step 4: Evaluate the Limit

Substitute $x = a$:

Numerator: $a \cdot a^{a-1} - a^a \ln a = a^a - a^a \ln a = a^a(1 - \ln a)$

Denominator: $a^a(\ln a + 1)$

So,

$$L = \frac{a^a(1 - \ln a)}{a^a(1 + \ln a)} = \frac{1 - \ln a}{1 + \ln a}$$

Final Answer:

$$\boxed{\frac{1 - \ln a}{1 + \ln a}}$$

6. $\lim_{x \to 0} \frac{e^x \sin x - x - x^2}{x^2 + x \log(1-x)}$

Answer:

Step 1: Check Indeterminate Form

Substitute $x = 0$:

Numerator: $e^0 \cdot \sin 0 - 0 - 0 = 1 \cdot 0 - 0 - 0 = 0$

Denominator: $0^2 + 0 \cdot \log(1) = 0 + 0 \cdot 0 = 0$

This is $\frac{0}{0}$. Apply L'Hospital's Rule.

Step 2: Apply L'Hospital's Rule (Differentiate Numerator and Denominator)

Numerator derivative: $\frac{d}{dx}[e^x \sin x - x - x^2] = e^x \sin x + e^x \cos x - 1 - 2x$

Denominator derivative: $\frac{d}{dx}[x^2 + x \log(1-x)] = 2x + [1 \cdot \log(1-x) + x \cdot \frac{-1}{1-x}] = 2x + \log(1-x) - \frac{x}{1-x}$

So,

$$L = \lim_{x \to 0} \frac{e^x \sin x + e^x \cos x - 1 - 2x}{2x + \log(1-x) - \frac{x}{1-x}}$$

Step 3: Check Form Again ($\frac{0}{0}$)

Substitute $x = 0$:

Numerator: $0 + 1 \cdot 1 - 1 - 0 = 1 - 1 = 0$

Denominator: $0 + \log(1) - 0 = 0 + 0 - 0 = 0$

Apply L'Hospital's Rule again.

Step 4: Apply L'Hospital's Rule Again

Differentiate numerator:

$$\frac{d}{dx}[e^x \sin x + e^x \cos x - 1 - 2x] = (e^x \sin x + e^x \cos x) + (e^x \cos x - e^x \sin x) - 2$$

$$= e^x \cos x + e^x \cos x - 2 = 2e^x \cos x - 2$$

$$\frac{d}{dx}[2x + \log(1-x) - \frac{x}{1-x}] = 2 + \frac{-1}{1-x} - \frac{(1)(1-x) - x(-1)}{(1-x)^2}$$

$$= 2 - \frac{1}{1-x} - \frac{1-x+x}{(1-x)^2} = 2 - \frac{1}{1-x} - \frac{1}{(1-x)^2}$$

Step 5: Evaluate the Limit

Substitute $x = 0$:

Numerator: $2e^0 \cos 0 - 2 = 2(1)(1) - 2 = 0$

Denominator: $2 - \frac{1}{1} - \frac{1}{1} = 2 - 1 - 1 = 0$

Still $\frac{0}{0}$. Apply L'Hospital's Rule a third time.

Step 6: Apply L'Hospital's Rule Third Time

Differentiate numerator:

$$\frac{d}{dx}[2e^x \cos x - 2] = 2e^x \cos x - 2e^x \sin x = 2e^x(\cos x - \sin x)$$

Differentiate denominator:

$$\frac{d}{dx}[2 - \frac{1}{1-x} - \frac{1}{(1-x)^2}] = 0 - \frac{(-1)}{(1-x)^2} - \frac{-2(-1)}{(1-x)^3} = \frac{1}{(1-x)^2} - \frac{2}{(1-x)^3}$$

(Using chain rule: $\frac{d}{dx}[(1-x)^{-2}] = -2(1-x)^{-3} \cdot (-1) = \frac{2}{(1-x)^3}$)

So,

$$L = \lim_{x \to 0} \frac{2e^x(\cos x - \sin x)}{\frac{1}{(1-x)^2} - \frac{2}{(1-x)^3}} = \lim_{x \to 0} \frac{2e^x(\cos x - \sin x)}{\frac{(1-x)-2}{(1-x)^3}} = \lim_{x \to 0} \frac{2e^x(\cos x - \sin x)(1-x)^3}{-1-x}$$

Step 7: Final Evaluation

Substitute $x = 0$:

$$L = \frac{2e^0(\cos 0 - \sin 0)(1-0)^3}{-1-0} = \frac{2(1)(1-0)(1)}{-1} = \frac{2}{-1} = -2$$

Final Answer:

$$\boxed{-2}$$

7. If $\lim_{x\to 0} \frac{\sin 2x + a\sin x}{x^3}$ be finite, find the value of a and hence limit.

Answer:

Step 1: Check Indeterminate Form

Substitute $x = 0$:

Numerator: $\sin 0 + a\sin 0 = 0 + 0 = 0$

Denominator: 0

$\frac{0}{0}$. For the limit to be finite, the numerator must approach zero at least as fast as x^3. This implies the coefficients of the lower powers of x in the numerator's series expansion must be zero.

Step 2: Expand Numerator Using Maclaurin Series

$$\sin 2x = 2x - \frac{(2x)^3}{3!} + \dots = 2x - \frac{8x^3}{6} + \dots = 2x - \frac{4x^3}{3} + \dots$$

$$a\sin x = a\left(x - \frac{x^3}{6} + \dots\right) = ax - \frac{ax^3}{6} + \dots$$

So, numerator:

$$\sin 2x + a\sin x = (2x + ax) + \left(-\frac{4x^3}{3} - \frac{ax^3}{6}\right) + \dots = (2+a)x + \left(-\frac{4}{3} - \frac{a}{6}\right)x^3 + \dots$$

Step 3: For Limit to be Finite, Coefficient of x^1 must be 0

$$2 + a = 0 \implies a = -2$$

Step 4: Find the Limit with $a = -2$

Substitute $a = -2$ into the coefficient of x^3:

$$-\frac{4}{3} - \frac{(-2)}{6} = -\frac{4}{3} + \frac{2}{6} = -\frac{4}{3} + \frac{1}{3} = -\frac{3}{3} = -1$$

So, the numerator behaves like $-1 \cdot x^3$ as $x \to 0$.

$$L = \lim_{x\to 0} \frac{-x^3 + \dots}{x^3} = -1$$

Final Answer:

$$\boxed{a = -2} \quad \text{and} \quad \boxed{\text{Limit} = -1}$$

8. If $\lim_{x\to 0} \frac{x(1 + a\cos x) - b\sin x}{x^3} = 1$, find the value of a and b.

Answer:

Step 1: Check Indeterminate Form and Apply Series Expansion

Numerator: $x(1 + a\cos x) - b\sin x$

Expand $\cos x$ and $\sin x$:

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + \ldots$$

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} + \ldots$$

So,

$$x(1 + a(1 - \frac{x^2}{2} + \ldots)) - b(x - \frac{x^3}{6} + \ldots) = x + ax - \frac{ax^3}{2} + \ldots - bx + \frac{bx^3}{6} + \ldots$$

$$= (1 + a - b)x + (-\frac{a}{2} + \frac{b}{6})x^3 + \ldots$$

Step 2: For Limit to Exist and Equal 1

The coefficient of x must be 0, and the coefficient of x^3 must be 1 (since denominator is x^3).

So,

1. $1 + a - b = 0$
2. $-\frac{a}{2} + \frac{b}{6} = 1$

Step 3: Solve the System of Equations

From (1): $b = 1 + a$

Substitute into (2):

$$-\frac{a}{2} + \frac{1+a}{6} = 1$$

Multiply both sides by 6:

$$-3a + (1 + a) = 6$$

$$-3a + 1 + a = 6$$

$$-2a = 5$$

$$a = -\frac{5}{2}$$

Then, $b = 1 + (-\frac{5}{2}) = -\frac{3}{2}$

Final Answer:

$$\boxed{a = -\frac{5}{2}} \quad \text{and} \quad \boxed{b = -\frac{3}{2}}$$

9. $\lim_{x\to 1} \left(\frac{x}{x-1} - \frac{1}{\log x}\right)$

Answer:

Step 1: Combine the Terms

$$L = \lim_{x\to 1} \left(\frac{x}{x-1} - \frac{1}{\log x}\right) = \lim_{x\to 1} \frac{x \log x - (x-1)}{(x-1)\log x}$$

Step 2: Check Indeterminate Form

Substitute $x = 1$:

Numerator: $1 \cdot \log 1 - (1-1) = 0 - 0 = 0$

Denominator: $(1-1)\log 1 = 0 \cdot 0 = 0$

This is $\frac{0}{0}$. Apply L'Hospital's Rule.

Step 3: Apply L'Hospital's Rule (Differentiate Numerator and Denominator)

Numerator: $f(x) = x\log x - x + 1$

$$f'(x) = \log x + x \cdot \frac{1}{x} - 1 = \log x + 1 - 1 = \log x$$

Denominator: $g(x) = (x-1)\log x$

$$g'(x) = (1)\log x + (x-1)\cdot\frac{1}{x} = \log x + \frac{x-1}{x}$$

So,

$$L = \lim_{x\to 1} \frac{\log x}{\log x + \frac{x-1}{x}}$$

Step 4: Check Form Again ($\frac{0}{0}$)

Substitute $x = 1$: Numerator: $\log 1 = 0$, Denominator: $0 + \frac{0}{1} = 0$.

Apply L'Hospital's Rule again.

Step 5: Apply L'Hospital's Rule Again

Differentiate numerator: $\frac{d}{dx}[\log x] = \frac{1}{x}$

Differentiate denominator: $\frac{d}{dx}[\log x + 1 - \frac{1}{x}] = \frac{1}{x} + \frac{1}{x^2}$

So,

$$L = \lim_{x\to 1} \frac{\frac{1}{x}}{\frac{1}{x} + \frac{1}{x^2}} = \lim_{x\to 1} \frac{\frac{1}{x}}{\frac{x+1}{x^2}} = \lim_{x\to 1} \frac{1}{x} \cdot \frac{x^2}{x+1} = \lim_{x\to 1} \frac{x}{x+1}$$

Step 6: Final Evaluation

$$L = \frac{1}{1+1} = \frac{1}{2}$$

Final Answer:

$$\boxed{\frac{1}{2}}$$

10. $\lim_{x\to 0}\left(\frac{1}{2x} - \frac{1}{x(e^{\pi x}+1)}\right)$

Answer:

Step 1: Combine the Terms

$$L = \lim_{x\to 0}\left(\frac{1}{2x} - \frac{1}{x(e^x+1)}\right) = \lim_{x\to 0}\frac{(e^x+1)-2}{2x(e^x+1)} = \lim_{x\to 0}\frac{e^x-1}{2x(e^x+1)}$$

Step 2: Check Indeterminate Form

Substitute $x = 0$: Numerator: $e^0 - 1 = 0$, Denominator: $2 \cdot 0 \cdot (1+1) = 0$.

$\frac{0}{0}$. Apply L'Hospital's Rule.

Step 3: Apply L'Hospital's Rule

Differentiate numerator: $\frac{d}{dx}[e^x - 1] = e^x$

Differentiate denominator: $\frac{d}{dx}[2x(e^x+1)] = 2[(1)(e^x+1) + x(e^x)] = 2(e^x + 1 + xe^x)$

So,

$$L = \lim_{x\to 0}\frac{e^x}{2(e^x+1+xe^x)}$$

Step 4: Evaluate the Limit

Substitute $x = 0$:

$$L = \frac{e^0}{2(e^0+1+0)} = \frac{1}{2(1+1)} = \frac{1}{4}$$

Final Answer:

$$\boxed{\frac{1}{4}}$$

11. $\lim_{x\to 0}\left(\frac{1}{x^2} - \cot^2 x\right)$

Answer:

Step 1: Rewrite in Terms of Sine and Cosine

$$\cot^2 x = \frac{\cos^2 x}{\sin^2 x}$$

So,

$$L = \lim_{x\to 0}\left(\frac{1}{x^2} - \frac{\cos^2 x}{\sin^2 x}\right) = \lim_{x\to 0}\frac{\sin^2 x - x^2\cos^2 x}{x^2\sin^2 x}$$

Step 2: Check Indeterminate Form

Substitute $x = 0$: Numerator: $0 - 0 = 0$, Denominator: $0 \cdot 0 = 0$.
$\frac{0}{0}$. Apply L'Hospital's Rule would be very messy. Use series expansion or algebraic manipulation.

Step 3: Use Algebraic Identity and Series Expansion

Note: $\sin^2 x - x^2 \cos^2 x = (\sin x - x \cos x)(\sin x + x \cos x)$
But a better approach is to use:

$$\frac{1}{x^2} - \frac{\cos^2 x}{\sin^2 x} = \frac{\sin^2 x - x^2 \cos^2 x}{x^2 \sin^2 x} = \frac{(\sin x - x \cos x)(\sin x + x \cos x)}{x^2 \sin^2 x}$$

Now, use series:

$$\sin x = x - \frac{x^3}{6} + \ldots, \quad \cos x = 1 - \frac{x^2}{2} + \ldots$$

Then,

$$\sin x - x \cos x = (x - \frac{x^3}{6}) - x(1 - \frac{x^2}{2}) + \ldots = x - \frac{x^3}{6} - x + \frac{x^3}{2} + \ldots = \frac{x^3}{3} + \ldots$$

$$\sin x + x \cos x = (x - \frac{x^3}{6}) + x(1 - \frac{x^2}{2}) + \ldots = x - \frac{x^3}{6} + x - \frac{x^3}{2} + \ldots = 2x - \frac{2x^3}{3} + \ldots$$

So, numerator $\approx \frac{x^3}{3} \cdot 2x = \frac{2x^4}{3}$
Denominator: $x^2 \cdot x^2 = x^4$
Thus,

$$L = \lim_{x \to 0} \frac{\frac{2x^4}{3}}{x^4} = \frac{2}{3}$$

Final Answer:

$$\boxed{\frac{2}{3}}$$

12. $\lim_{x \to 0} \left(\frac{e^x + e^{2x} + e^{3x}}{3} \right)^{1/x}$

Answer:

Step 1: Recognize 1^∞ Indeterminate Form. Take Logarithm.

Let $L = \lim_{x\to 0} \left(\frac{e^x+e^{2x}+e^{3x}}{3}\right)^{1/x}$

Then,

$$\ln L = \lim_{x\to 0} \frac{1}{x} \ln\left(\frac{e^x+e^{2x}+e^{3x}}{3}\right) = \lim_{x\to 0} \frac{\ln(e^x+e^{2x}+e^{3x}) - \ln 3}{x}$$

Step 2: Check Indeterminate Form ($\frac{0}{0}$)

Apply L'Hospital's Rule:

$$\ln L = \lim_{x\to 0} \frac{\frac{d}{dx}[\ln(e^x+e^{2x}+e^{3x})]}{1} = \lim_{x\to 0} \frac{e^x+2e^{2x}+3e^{3x}}{e^x+e^{2x}+e^{3x}}$$

Step 3: Evaluate the Limit

Substitute $x = 0$:

$$\ln L = \frac{1+2+3}{1+1+1} = \frac{6}{3} = 2$$

So, $L = e^2$

Final Answer:

$$\boxed{e^2}$$

13. $\lim_{x\to 0} \left(\frac{\tan x}{x}\right)^{1/x^2}$

Answer:

Step 1: Recognize 1^∞ Form. Take Logarithm.

Let $L = \lim_{x\to 0} \left(\frac{\tan x}{x}\right)^{1/x^2}$

$$\ln L = \lim_{x\to 0} \frac{1}{x^2} \ln\left(\frac{\tan x}{x}\right) = \lim_{x\to 0} \frac{\ln(\tan x) - \ln x}{x^2}$$

Step 2: Check Indeterminate Form ($\frac{0}{0}$)

Apply L'Hospital's Rule:

$$\ln L = \lim_{x\to 0} \frac{\frac{1}{\tan x}\sec^2 x - \frac{1}{x}}{2x} = \lim_{x\to 0} \frac{\frac{\sec^2 x}{\tan x} - \frac{1}{x}}{2x} = \lim_{x\to 0} \frac{\frac{1}{\sin x \cos x} - \frac{1}{x}}{2x}$$

$$= \lim_{x\to 0} \frac{\frac{x - \sin x \cos x}{x \sin x \cos x}}{2x} = \lim_{x\to 0} \frac{x - \sin x \cos x}{2x^2 \sin x \cos x}$$

Step 3: Simplify and Use Series Expansion

Note: $\sin x \cos x = \frac{1}{2}\sin 2x$

But use series:

$$\sin x = x - \frac{x^3}{6} + ..., \quad \cos x = 1 - \frac{x^2}{2} + ...$$

So,

$$\sin x \cos x = (x - \frac{x^3}{6})(1 - \frac{x^2}{2}) + ... = x - \frac{x^3}{2} - \frac{x^3}{6} + ... = x - \frac{2x^3}{3} + ...$$

Then,

$$x - \sin x \cos x = x - (x - \frac{2x^3}{3}) = \frac{2x^3}{3}$$

Denominator: $2x^2 \cdot x \cdot 1 = 2x^3$

Thus,

$$\ln L = \lim_{x \to 0} \frac{\frac{2x^3}{3}}{2x^3} = \frac{1}{3}$$

So, $L = e^{1/3}$

Final Answer:

$$\boxed{e^{1/3}}$$

14. $\lim_{x \to 0} (\cos x)^{\cot x}$

Answer:

Step 1: Recognize 1^∞ Form. Take Logarithm.

Let $L = \lim_{x \to 0} (\cos x)^{\cot x}$

$$\ln L = \lim_{x \to 0} \cot x \cdot \ln(\cos x) = \lim_{x \to 0} \frac{\cos x}{\sin x} \cdot \ln(\cos x)$$

Step 2: Rewrite and Check Indeterminate Form

$$\ln L = \lim_{x \to 0} \frac{\ln(\cos x)}{\tan x}$$

Substitute $x = 0$: Numerator: $\ln(1) = 0$, Denominator: $\tan 0 = 0$. $\frac{0}{0}$. Apply L'Hospital's Rule.

Step 3: Apply L'Hospital's Rule

$$\ln L = \lim_{x \to 0} \frac{\frac{1}{\cos x} \cdot (-\sin x)}{\sec^2 x} = \lim_{x \to 0} \frac{-\tan x}{\sec^2 x} = \lim_{x \to 0} -\tan x \cos^2 x = \lim_{x \to 0} -\sin x \cos x$$

$$= -0 \cdot 1 = 0$$

So, $L = e^0 = 1$

Final Answer:

$$\boxed{1}$$

15. $\lim_{x \to \infty} \left(a^{1/x} - 1\right)^x$

Answer:

Step 1: Let $t = 1/x$, so $t \to 0^+$

$$L = \lim_{t \to 0^+} (a^t - 1)^{1/t}$$

Step 2: Recognize 0^0 Form. Take Logarithm.

$$\ln L = \lim_{t \to 0^+} \frac{1}{t} \ln(a^t - 1)$$

Step 3: Check Indeterminate Form ($\infty \cdot -\infty$), Rewrite

$$\ln L = \lim_{t \to 0^+} \frac{\ln(a^t - 1)}{t}$$

Substitute $t = 0$: Numerator: $\ln(0) \to -\infty$, Denominator: 0. This is not $\frac{0}{0}$ but $\frac{-\infty}{0}$. Use substitution.

Step 4: Let $u = a^t - 1$, then as $t \to 0, u \to 0$, and $t = \frac{\ln(u+1)}{\ln a}$
Then,

$$\ln L = \lim_{u \to 0} \frac{\ln u}{\frac{\ln(u+1)}{\ln a}} = \ln a \cdot \lim_{u \to 0} \frac{\ln u}{\ln(u+1)}$$

Now, $\ln(u+1) \sim u$ as $u \to 0$, so:

$$\ln L = \ln a \cdot \lim_{u \to 0} \frac{\ln u}{u} = \ln a \cdot (-\infty) = -\infty$$

Thus, $L = e^{-\infty} = 0$
Final Answer:

$$\boxed{0}$$