# Term Work Assignment- 2 Solution

**Subject Name & Code:**

*Mathematics- I - BE01R00041*

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## TWA-2: Taylor and Maclaurin series

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### Part 1: Elementary Series

1. **Maclaurin Series Expansion: Obtain the Maclaurin series for the following functions by finding the general form for the derivatives:**

The Maclaurin series for a function $f(x)$ is given by:
$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$

- $\boldsymbol{f(x) = \sin(x)}$

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**Answer:**

- $f(0) = \sin(0) = 0$
- $f'(x) = \cos(x) \Rightarrow f'(0) = 1$
- $f''(x) = -\sin(x) \Rightarrow f''(0) = 0$
- $f'''(x) = -\cos(x) \Rightarrow f'''(0) = -1$
- $f^{(4)}(x) = \sin(x) \Rightarrow f^{(4)}(0) = 0$
- $f^{(5)}(x) = \cos(x) \Rightarrow f^{(5)}(0) = 1$

  The pattern of derivatives at $x = 0$ is: $0, 1, 0, -1, 0, 1, 0, -1, \ldots$

Only the odd-powered terms are non-zero. The general form is:
$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}$

- $\boldsymbol{f(x) = \cos(x)}$

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**Answer:**

- $f(0) = \cos(0) = 1$
- $f'(x) = -\sin(x) \Rightarrow f'(0) = 0$
- $f''(x) = -\cos(x) \Rightarrow f''(0) = -1$

- $f'''(x) = \sin(x) \Rightarrow f'''(0) = 0$
- $f^{(4)}(x) = \cos(x) \Rightarrow f^{(4)}(0) = 1$

  The pattern of derivatives at $x = 0$ is: $1, 0, -1, 0, 1, 0, -1, \ldots$

Only the even-powered terms are non-zero. The general form is:
$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$

- $f(x) = e^x \log(1 + x)$

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**Answer:**

This requires finding a pattern for the nth derivative or using the series multiplication of known series.

- Known Maclaurin series:
  - $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$
  - $\log(1 + x) = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} x^m = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

The product $f(x) = e^x \cdot \log(1 + x)$ is given by the Cauchy product of these two series:
$f(x) = \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right) \cdot \left(\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} x^m\right)$

To find the coefficient $a_k$ for $x^k$, we sum over all pairs $n, m$ such that $n + m = k$ and $m \geq 1$:
$a_k = \sum_{m=1}^{k} \frac{1}{(k-m)!} \cdot \frac{(-1)^{m+1}}{m}$
The first few terms are:
$f(x) = (1)(x) + (1)(-\frac{x^2}{2}) + (1)(\frac{x^3}{3}) + (x)(x) + (x)(-\frac{x^2}{2}) + (\frac{x^2}{2!})(x) + \cdots$
Let's carefully combine terms up to $x^4$:

- $x^1$ coeff: $1 \cdot 1 = 1$
- $x^2$ coeff: $1 \cdot (-\frac{1}{2}) + 1 \cdot 1 = -\frac{1}{2} + 1 = \frac{1}{2}$
- $x^3$ coeff: $1 \cdot (\frac{1}{3}) + 1 \cdot (-\frac{1}{2}) + \frac{1}{2!} \cdot 1 = \frac{1}{3} - \frac{1}{2} + \frac{1}{2} = \frac{1}{3}$
- $x^4$ coeff: $1 \cdot (-\frac{1}{4}) + 1 \cdot (\frac{1}{3}) + \frac{1}{2!} \cdot (-\frac{1}{2}) + \frac{1}{3!} \cdot 1 = -\frac{1}{4} + \frac{1}{3} - \frac{1}{4} + \frac{1}{6}$

  Find a common denominator (12): $= \frac{-3}{12} + \frac{4}{12} - \frac{3}{12} + \frac{2}{12} = 0$

Thus, the Maclaurin series is:
$e^x \log(1 + x) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + 0 \cdot x^4 + \cdots$
*(The next non-zero term would be found by continuing this process for higher powers)*

### 2. Using Known Series:

- Obtain the Maclaurin series for $\log\left(\frac{1+x}{1-x}\right)$ by using the series for $\log(1+x)$ and $\log(1-x)$.

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**Answer:**

Known series:

- $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots$
- $\log(1-x) = (-x) - \frac{(-x)^2}{2} + \frac{(-x)^3}{3} - \frac{(-x)^4}{4} + \cdots = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$

Subtract the second series from the first:

$\log(1+x) - \log(1-x) = \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right) -$
$\left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots\right)$
$= x + x - \frac{x^2}{2} + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^4}{4} + \cdots$
$= 2x + 0 + \frac{2x^3}{3} + 0 + \frac{2x^5}{5} + \cdots$

Therefore, the series is:

$\log\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots\right) = 2\sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}$

- Obtain the Maclaurin series for $\cos(x^2)$.

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**Answer:**

This is a simple substitution. We know:

$\cos(u) = 1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \frac{u^6}{6!} + \cdots$

Now substitute $u = x^2$:

$\cos(x^2) = 1 - \frac{(x^2)^2}{2!} + \frac{(x^2)^4}{4!} - \frac{(x^2)^6}{6!} + \cdots$

$\cos(x^2) = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{4n}$

## Part 2: Taylor Series and Approximation

The Taylor series for a function $f(x)$ about a point $x = a$ is:

$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$

### 1. Taylor Expansion:

- Expand the function $f(x) = \tan(x)$ in powers of $(x - \frac{\pi}{4})$. Show at least four non-zero terms.

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**Answer:**

We need to find derivatives of $\tan(x)$ and evaluate them at $a = \pi/4$.

- $f(x) = \tan(x) \Rightarrow f(\pi/4) = 1$
- $f'(x) = \sec^2(x) \Rightarrow f'(\pi/4) = (\sqrt{2})^2 = 2$
- $f''(x) = 2\sec^2(x)\tan(x) \Rightarrow f''(\pi/4) = 2 \cdot 2 \cdot 1 = 4$
- $f'''(x) = 2[\sec^2(x)\sec^2(x) + \tan(x) \cdot 2\sec^2(x)\tan(x)]$ (Using Product Rule on $f''(x)$)
  $= 2\sec^4(x) + 4\sec^2(x)\tan^2(x)$
  Evaluate at $\pi/4$: $= 2(\sqrt{2})^4 + 4(\sqrt{2})^2(1)^2 = 2 \cdot 4 + 4 \cdot 2 \cdot 1 = 8 + 8 = 16$

Now, let $h = (x - \pi/4)$. The Taylor series is:
$\tan(x) = f(\pi/4) + f'(\pi/4)h + \frac{f''(\pi/4)}{2!}h^2 + \frac{f'''(\pi/4)}{3!}h^3 + \cdots$
$\tan(x) = 1 + 2h + \frac{4}{2}h^2 + \frac{16}{6}h^3 + \cdots$
$\tan(x) = 1 + 2(x - \frac{\pi}{4}) + 2(x - \frac{\pi}{4})^2 + \frac{8}{3}(x - \frac{\pi}{4})^3 + \cdots$

- Expand $\sin(\frac{\pi}{4} + x)$ in powers of $x$.

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**Answer:**

This is a Maclaurin series for $f(u) = \sin(u)$ where $u = \frac{\pi}{4} + x$, but expanded around $x = 0$ (i.e., $u = \pi/4$). We find derivatives of $f(x) = \sin(\pi/4 + x)$ with respect to $x$ and evaluate them at $x = 0$.

- $f(x) = \sin(\frac{\pi}{4} + x) \Rightarrow f(0) = \sin(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$
- $f'(x) = \cos(\frac{\pi}{4} + x) \Rightarrow f'(0) = \cos(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$
- $f''(x) = -\sin(\frac{\pi}{4} + x) \Rightarrow f''(0) = -\sin(\frac{\pi}{4}) = -\frac{1}{\sqrt{2}}$
- $f'''(x) = -\cos(\frac{\pi}{4} + x) \Rightarrow f'''(0) = -\cos(\frac{\pi}{4}) = -\frac{1}{\sqrt{2}}$
- $f^{(4)}(x) = \sin(\frac{\pi}{4} + x) \Rightarrow f^{(4)}(0) = \sin(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$

The pattern of derivatives at $x = 0$ is: $\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \ldots$

The Maclaurin series is:
$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots$
$\sin(\frac{\pi}{4} + x) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}x + \frac{(-1/\sqrt{2})}{2}x^2 + \frac{(-1/\sqrt{2})}{6}x^3 + \frac{(1/\sqrt{2})}{24}x^4 + \cdots$
$\sin(\frac{\pi}{4} + x) = \frac{1}{\sqrt{2}}\left(1 + x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right)$

This matches the result obtained using the trigonometric identity.

## 2. Approximation with Taylor's Theorem:

- Use the Taylor expansion of $\sin(\frac{\pi}{4}+x)$ to find the approximate value of $\sin(44°)$.

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**Answer:**

First, convert the angle to radians because our series uses radians.

$44° = 45° - 1° = \frac{\pi}{4} - \frac{\pi}{180}$ radians.

Therefore, $x = -\frac{\pi}{180} \approx -0.0174533$.

We will use the series we just derived up to the $x^2$ term:

$\sin(\frac{\pi}{4}+x) \approx \frac{1}{\sqrt{2}}\left(1 + x - \frac{x^2}{2}\right)$

Now substitute $x = -\frac{\pi}{180}$:

$\sin(44°) \approx \frac{1}{\sqrt{2}}\left(1 + (-\frac{\pi}{180}) - \frac{(-\frac{\pi}{180})^2}{2}\right)$

$= \frac{1}{\sqrt{2}}\left(1 - \frac{\pi}{180} - \frac{\pi^2}{2\cdot 180^2}\right)$

Using $\pi \approx 3.141593$:

- $\frac{\pi}{180} \approx 0.0174533$
- $\frac{\pi^2}{2\cdot 32400} = \frac{9.869604}{64800} \approx 0.0001523$

Now calculate the value inside the parentheses:

$1 - 0.0174533 - 0.0001523 = 0.9823944$

Finally, multiply by $1/\sqrt{2} \approx 0.70710678$:

$\sin(44°) \approx 0.70710678 \times 0.9823944 \approx 0.694658$

(The actual value is approximately $\sin(44°) \approx 0.694658$, confirming the accuracy).

- Use Taylor's Theorem to find the approximate value of $\sqrt{25.15}$.

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**Answer:**

We use the Taylor series for $f(x) = \sqrt{x}$ around a perfect square, $a = 25$.

Let $f(x) = x^{1/2}$.

- $f(25) = 5$
- $f'(x) = \frac{1}{2}x^{-1/2} \Rightarrow f'(25) = \frac{1}{2\cdot 5} = 0.1$
- $f''(x) = -\frac{1}{4}x^{-3/2} \Rightarrow f''(25) = -\frac{1}{4\cdot 125} = -0.002$

The Taylor series is:
$f(x) \approx f(25) + f'(25)(x-25) + \frac{f''(25)}{2!}(x-25)^2$
$\sqrt{x} \approx 5 + 0.1(x-25) + \frac{-0.002}{2}(x-25)^2$
$\sqrt{x} \approx 5 + 0.1(x-25) - 0.001(x-25)^2$

Now, for $x = 25.15$, $x - 25 = 0.15$:
$\sqrt{25.15} \approx 5 + 0.1(0.15) - 0.001(0.15)^2$
$= 5 + 0.015 - 0.001(0.0225)$
$= 5.015 - 0.0000225$
$\sqrt{25.15} \approx 5.0149775$
(The actual value is approximately $5.014975$, which is very close).

- Given $\log_{10} 73 = 1.8633$ and $\log_{10} e = 0.4343$, use Taylor's Theorem to find the approximate value of $\log_{10} 73.55$.

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**Answer:**

We use the Taylor series for $f(x) = \log_{10} x$ around $a = 73$.
Note: The derivative of $\log_{10} x$ is $\frac{1}{x}\log_{10} e$.

- $f(73) = 1.8633$
- $f'(x) = \frac{\log_{10} e}{x} \Rightarrow f'(73) = \frac{0.4343}{73} \approx 0.0059493$

The first-order Taylor approximation is:
$f(x) \approx f(73) + f'(73)(x-73)$
$\log_{10} x \approx 1.8633 + 0.0059493(x-73)$

For $x = 73.55$, $x - 73 = 0.55$:
$\log_{10} 73.55 \approx 1.8633 + 0.0059493(0.55)$
$= 1.8633 + 0.0032721$
$\log_{10} 73.55 \approx 1.8665721$

- Find the approximate value of $\tan^{-1}(1.003)$ using Taylor's Theorem. Use $\pi = 3.141593$.

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**Answer:**
We use the Taylor series for $f(x) = \tan^{-1}(x)$ around $a = 1$, as it is a nice point where we know $f(1) = \pi/4$.
Let $f(x) = \tan^{-1}(x)$.

- $f(1) = \pi/4 \approx 0.78539825$
- $f'(x) = \frac{1}{1+x^2} \Rightarrow f'(1) = \frac{1}{1+1} = 0.5$
- $f''(x) = -\frac{2x}{(1+x^2)^2} \Rightarrow f''(1) = -\frac{2}{(4)} = -0.5$
- $f'''(x) = \frac{d}{dx}\left(-\frac{2x}{(1+x^2)^2}\right)$. Using the quotient rule, this evaluates to a manageable number, but let's use a first-order approximation for a very small change $(x - 1 = 0.003)$.

First-order approximation:
$f(x) \approx f(1) + f'(1)(x-1)$
$\tan^{-1}(x) \approx \frac{\pi}{4} + 0.5(x-1)$

For $x = 1.003$, $x - 1 = 0.003$:
$\tan^{-1}(1.003) \approx 0.78539825 + 0.5(0.003)$
$= 0.78539825 + 0.0015$
$\tan^{-1}(1.003) \approx 0.78689825$

## Part 3: Advanced Problems and Applications

1. **Polynomial Expansion:** Arrange the polynomial $f(x) = x^3 - 3x^2 + 4x + 3$ in powers of $(x - 2)$.

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**Answer:**
This is finding the Taylor series of the polynomial about $a = 2$. We find derivatives.

- $f(x) = x^3 - 3x^2 + 4x + 3 \Rightarrow f(2) = 8 - 12 + 8 + 3 = 7$
- $f'(x) = 3x^2 - 6x + 4 \Rightarrow f'(2) = 12 - 12 + 4 = 4$
- $f''(x) = 6x - 6 \Rightarrow f''(2) = 12 - 6 = 6$
- $f'''(x) = 6 \Rightarrow f'''(2) = 6$
- All higher derivatives are 0.

The Taylor series is:
$f(x) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3$
$f(x) = 7 + 4(x-2) + \frac{6}{2}(x-2)^2 + \frac{6}{6}(x-2)^3$
$f(x) = 7 + 4(x-2) + 3(x-2)^2 + 1(x-2)^3$

Thus, the polynomial in powers of $(x-2)$ is:
$f(x) = (x-2)^3 + 3(x-2)^2 + 4(x-2) + 7$

2. **Series for Integrals:** Obtain the Maclaurin series of $\int \frac{\sin x}{x} dx$.

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**Answer:**

The function $\frac{\sin x}{x}$ is known as the **sinc function**. Its integral, $\int \frac{\sin x}{x} dx$, is a very important function in mathematics and engineering, known as the **Sine Integral**, denoted by $\text{Si}(x)$. Its value at $x = 0$ is defined by the limit to be 0, i.e., $\text{Si}(0) = 0$.

**Step 1: Find the Maclaurin Series for the Integrand $\frac{\sin x}{x}$**

We start with the known Maclaurin series for $\sin x$:

$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

Now, we divide this entire series by $x$ (for $x \neq 0$):

$\frac{\sin x}{x} = \frac{1}{x}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$

$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$

This can be written in summation form as:

$\frac{\sin x}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n}$

**Step 2: Integrate the Series Term-by-Term**

A key property of power series is that they can be integrated term-by-term within their radius of convergence. The series for $\frac{\sin x}{x}$ converges for all $x$.

Integrating the series representation:

$\int \frac{\sin x}{x} dx = \int \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n}\right) dx$

Because the series converges uniformly on any closed interval, we can interchange the sum and the integral:

$\int \frac{\sin x}{x} dx = \sum_{n=0}^{\infty} \int \frac{(-1)^n}{(2n+1)!} x^{2n} dx$

$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \int x^{2n} dx$

$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot \frac{x^{2n+1}}{2n+1} + C$

$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+1)!} x^{2n+1} + C$

**Step 3: Determine the Constant of Integration $C$**

The Sine Integral is defined such that $\text{Si}(0) = 0$. Let's evaluate our series at $x = 0$:

$\text{Si}(0) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+1)!} (0)^{2n+1} + C = 0 + C$

Therefore, to satisfy $\text{Si}(0) = 0$, we must have $C = 0$.

**Step 4: Final Maclaurin Series**

The Maclaurin series for the Sine Integral, $\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t} dt$, is:

$$\boxed{\int \frac{\sin x}{x} dx = \mathrm{Si}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+1)!} x^{2n+1}}$$

Writing out the first few terms explicitly:

$\mathrm{Si}(x) = x - \frac{x^3}{3 \cdot 3!} + \frac{x^5}{5 \cdot 5!} - \frac{x^7}{7 \cdot 7!} + \cdots$
$= x - \frac{x^3}{3 \cdot 6} + \frac{x^5}{5 \cdot 120} - \frac{x^7}{7 \cdot 5040} + \cdots$

$$\boxed{\mathrm{Si}(x) = x - \frac{x^3}{18} + \frac{x^5}{600} - \frac{x^7}{35280} + \frac{x^9}{3265920} - \cdots}$$

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