

## Term Work Assignment- 4 Solution

**Subject Name & Code:**

*Mathematics- I - BE01R00041*

### TWA-4: Applications of Partial Derivatives

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**Problem 1 (Tangent plane).** Find the equation of the tangent plane to the surface  $z = x^2 + xy + 2y^2$  at the point  $(1, -1, f(1, -1))$ .

**Answer:**

**Step 1:** Find the z-coordinate of the point.

$$f(1, -1) = (1)^2 + (1)(-1) + 2(-1)^2 = 1 - 1 + 2 = 2$$

So the point is  $(1, -1, 2)$ .

**Step 2:** The formula for the tangent plane to a surface  $z=f(x,y)$  at point  $(x_0,y_0,z_0)$  is:

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

**Step 3:** Find the partial derivatives.

$$f_x = \frac{\partial}{\partial x}(x^2 + xy + 2y^2) = 2x + y$$

$$f_y = \frac{\partial}{\partial y}(x^2 + xy + 2y^2) = x + 4y$$

**Step 4:** Evaluate the partial derivatives at the point  $(1, -1)$ .

$$f_x(1, -1) = 2(1) + (-1) = 2 - 1 = 1$$

$$f_y(1, -1) = (1) + 4(-1) = 1 - 4 = -3$$

**Step 5:** Plug all values into the tangent plane formula.

$$z - 2 = (1)(x - 1) + (-3)(y - (-1))$$

$$z - 2 = (x - 1) - 3(y + 1)$$

**Step 6:** Simplify the equation.

$$z - 2 = x - 1 - 3y - 3$$

$$z = x - 3y - 2$$

$$\boxed{x - 3y - z = 2} \text{ (This is the final answer in standard form.)}$$

**Problem 2 (Steepest ascent).** For  $f(x, y) = x^2 + 3y^2 - 4x + 12y$ , find the direction of maximal increase and the maximal directional derivative at  $(1, -2)$ .

**Answer:**

**Concept:** The direction of steepest ascent is given by the gradient vector  $\nabla f$ . The maximum value of the directional derivative is the magnitude of the gradient.

**Step 1:** Find the gradient  $\nabla f = \langle f_x, f_y \rangle$ .

$$f_x = \frac{\partial}{\partial x}(x^2 + 3y^2 - 4x + 12y) = 2x - 4$$

$$f_y = \frac{\partial}{\partial y}(x^2 + 3y^2 - 4x + 12y) = 6y + 12$$

**Step 2:** Evaluate the gradient at the point  $(1, -2)$ .

$$f_x(1, -2) = 2(1) - 4 = 2 - 4 = -2$$

$$f_y(1, -2) = 6(-2) + 12 = -12 + 12 = 0$$

$$\text{So, } \nabla f(1, -2) = \langle -2, 0 \rangle$$

**Step 3:** Find the direction of maximal increase. This is the unit vector in the direction of  $\nabla f$ .

$$\text{Direction} = \frac{\nabla f}{\|\nabla f\|} = \frac{\langle -2, 0 \rangle}{\sqrt{(-2)^2 + 0^2}} = \frac{\langle -2, 0 \rangle}{2} = \langle -1, 0 \rangle$$

This is a unit vector pointing directly left along the x-axis.

**Step 4:** Find the maximal directional derivative. This is  $\|\nabla f(1, -2)\|$ .

$$\|\langle -2, 0 \rangle\| = \sqrt{(-2)^2 + 0^2} = \sqrt{4} = 2$$

**Final Answers:**

- Direction:  $\langle -1, 0 \rangle$
- Max Derivative:  $2$

**Problem 3 (Critical points and classification).** Find and classify all critical points of  $f(x, y) = x^3 - 3xy^2$ .

**Answer:**

**Step 1:** Find the first partial derivatives and set them equal to zero to find critical points.

$$f_x = \frac{\partial}{\partial x}(x^3 - 3xy^2) = 3x^2 - 3y^2$$

$$f_y = \frac{\partial}{\partial y}(x^3 - 3xy^2) = 0 - 6xy = -6xy$$

Set  $f_x=0$  and  $f_y=0$ :

$$\begin{cases} 3x^2 - 3y^2 = 0 & (1) \\ -6xy = 0 & (2) \end{cases}$$

**Step 2:** Solve the system of equations.

From equation (2):  $-6xy=0 \Rightarrow$  either  $x=0$  or  $y=0$ .

- **Case 1:**  $x = 0$ . Substitute into equation (1):  
 $3(0)^2 - 3y^2 = 0 \implies -3y^2 = 0 \implies y = 0$   
 Critical Point:  $(0, 0)$

- **Case 2:**  $y = 0$ . Substitute into equation (1):  
 $3x^2 - 3(0)^2 = 0 \implies 3x^2 = 0 \implies x = 0$   
 This gives the same point,  $(0, 0)$ .

It appears we only have one critical point:  $(0, 0)$ .

**Step 3:** Classify the critical point using the Second Derivative Test (D-Test).

Find the second partial derivatives:

$$f_{xx} = \frac{\partial}{\partial x}(3x^2 - 3y^2) = 6x$$

$$f_{yy} = \frac{\partial}{\partial y}(-6xy) = -6x$$

$$f_{xy} = \frac{\partial}{\partial x}(-6xy) = -6y$$

Now, the discriminant  $D$  is given by:

$$D(x, y) = f_{xx}f_{yy} - (f_{xy})^2$$

Evaluate  $D$  at the critical point  $(0, 0)$ :

$$f_{xx}(0, 0) = 6(0) = 0$$

$$f_{yy}(0, 0) = -6(0) = 0$$

$$f_{xy}(0, 0) = -6(0) = 0$$

$$D(0, 0) = (0)(0) - (0)^2 = 0$$

Since  $D=0$ , the Second Derivative Test is **inconclusive**. We cannot classify the point  $(0, 0)$  as a max, min, or saddle point using this test. Further analysis (e.g., looking at paths approaching the point) is needed, which is likely beyond the scope of this problem. The answer is that the only critical point is  $(0, 0)$  and it cannot be classified by the D-test.

**Problem 4 (Laplace's equation).** Suppose  $u$  has continuous second partials and satisfies  $u_{xx} + u_{yy} = 0$ . If  $u_{xx}(1, 2) = 4$ , find  $u_{yy}(1, 2)$ .

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**Answer:**

**Concept:** Laplace's equation is  $\nabla^2 u = u_{xx} + u_{yy} = 0$ . This holds for all points  $(x, y)$  where the function is defined.

**Step 1:** Apply Laplace's equation at the point  $(1, 2)$ .

$$u_{xx}(1, 2) + u_{yy}(1, 2) = 0$$

**Step 2:** Substitute the known value and solve.

$$4 + u_{yy}(1, 2) = 0$$

$$u_{yy}(1, 2) = -4$$

**Final Answer:**

$$\boxed{-4}$$

**Problem 5 (Lagrange on a line).** Use Lagrange multipliers to find the extrema of  $f(x, y) = x^2 + y^2$  subject to the constraint  $x + y = 1$ .

**Answer:**

**Concept:** We want to optimize  $f(x, y)$  subject to  $g(x, y) = x + y - 1 = 0$ . The method of Lagrange multipliers states that at an extremum,  $\nabla f = \lambda \nabla g$ .

**Step 1:** Set up the equations.

$$\nabla f = \lambda \nabla g$$

$$\langle 2x, 2y \rangle = \lambda \langle 1, 1 \rangle$$

This gives us the system:

$$\begin{cases} 2x = \lambda & (1) \\ 2y = \lambda & (2) \\ x + y = 1 & (3) \end{cases}$$

**Step 2:** Solve the system.

From equations (1) and (2):  $2x = \lambda$  and  $2y = \lambda$ , so  $2x = 2y \implies x = y$ .

Substitute  $x=y$  into the constraint equation (3):

$$y + y = 1 \implies 2y = 1 \implies y = \frac{1}{2}$$

Since  $x = y$ ,  $x = \frac{1}{2}$ .

The only candidate point is  $(1/2, 1/2)$ .

**Step 3:** Find the value of  $f$  at this point.

$$f\left(\frac{1}{2}, \frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

**Step 4:** Justify that this is a minimum. Since the function  $f(x, y) = x^2 + y^2$  represents the squared distance from the origin, and the constraint is a line, the point we found is clearly the closest point on the line to the origin. There is no maximum distance from a point to a line, so this single extremum must be the **minimum**.

**Final Answer:** The function  $f$  has a **minimum** value of  $1/2$  at the point  $(1/2, 1/2)$ .

**Problem 6 (Normal vector and normal line).** For the surface  $z = xy^2$ , find (i) a normal vector and (ii) parametric equations of the normal line at  $(2, 1, 2)$ .

**Answer:**

**Concept:** For a surface defined explicitly as  $z = f(x, y)$ , we can rewrite it as a level surface:

$F(x, y, z) = f(x, y) - z = 0$ . The gradient  $\nabla F$  is a normal vector to the surface.

**Step 1:** Define the function  $F(x,y,z)$ .

$$F(x, y, z) = xy^2 - z$$

**Step 2:** Find the gradient  $\nabla F = \langle F_x, F_y, F_z \rangle$ , which is the normal vector.

$$F_x = \frac{\partial}{\partial x}(xy^2 - z) = y^2$$

$$F_y = \frac{\partial}{\partial y}(xy^2 - z) = 2xy$$

$$F_z = \frac{\partial}{\partial z}(xy^2 - z) = -1$$

So, a general normal vector is  $\nabla F = \langle y^2, 2xy, -1 \rangle$

**Step 3:** Evaluate the normal vector at the point  $(2,1,2)$ .

$$\nabla F(2, 1, 2) = \langle (1)^2, 2(2)(1), -1 \rangle = \langle 1, 4, -1 \rangle$$

**(i) Final Answer for Normal Vector:**  $\langle 1, 4, -1 \rangle$

**Step 4:** Find the parametric equations of the normal line.

The normal line passes through  $(x_0, y_0, z_0) = (2, 1, 2)$  and is parallel to the direction vector  $\vec{v} = \nabla F = \langle 1, 4, -1 \rangle$ .

The parametric equations are:

$$x = x_0 + v_x t = 2 + 1 \cdot t$$

$$y = y_0 + v_y t = 1 + 4 \cdot t$$

$$z = z_0 + v_z t = 2 + (-1) \cdot t$$

**(ii) Final Answer for Normal Line:**

$$x = 2 + t$$

$$y = 1 + 4t$$

$$z = 2 - t$$

**Problem 7 (Differentials error estimate).** *A rectangle has measured sides  $x = 10$ , cm,  $y = 5$ , cm with errors  $\Delta x = 0.1$ , cm,  $\Delta y = -0.05$ , cm. Using differentials, estimate the change in area  $A = xy$ .*

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**Answer:**

**Concept:** The differential  $dA$  approximates the change  $\Delta A$ .

$$dA = \frac{\partial A}{\partial x} dx + \frac{\partial A}{\partial y} dy \approx \Delta A$$

**Step 1:** Find the partial derivatives of  $A = xy$ .

$$\frac{\partial A}{\partial x} = y$$

$$\frac{\partial A}{\partial y} = x$$

**Step 2:** Substitute the values  $x=10, y=5, dx=\Delta x=0.1, dy=\Delta y=-0.05$  into the differential formula.

$$dA = (y)(dx) + (x)(dy) = (5)(0.1) + (10)(-0.05)$$

$$dA = 0.5 - 0.5$$

$$dA = 0\text{cm}^2$$

**Interpretation:** The estimated change in area is  $0\text{ cm}^2$ . This makes sense because a positive error in one side is canceled out by a negative error in the other side, given the specific values. The first-order estimate suggests no net change.

**Final Answer:**

$$\boxed{0\text{cm}^2}$$

**Problem 8 (Tangent plane: level surface). Find the tangent plane and the normal line to the surface  $x^2 + y^2 + z - 9 = 0$  at the point  $(1,2,4)$ .**

**Answer:**

**Concept:** For a level surface  $F(x, y, z) = 0$ , the tangent plane at  $(x_0, y_0, z_0)$  is:

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

The normal line uses  $\nabla F$  as its direction vector.

**Step 1:** Define  $F(x,y,z)$ .

$$F(x, y, z) = x^2 + y^2 + z - 9$$

**Step 2:** Find the gradient  $\nabla F$ .

$$F_x = 2x$$

$$F_y = 2y$$

$$F_z = 1$$

**Step 3:** Evaluate  $\nabla F$  at the point  $(1,2,4)$ .

$$\nabla F(1, 2, 4) = \langle 2(1), 2(2), 1 \rangle = \langle 2, 4, 1 \rangle$$

**Step 4:** Find the equation of the tangent plane.

Using the point and the normal vector components:

$$2(x - 1) + 4(y - 2) + 1(z - 4) = 0$$

Simplify:

$$2x - 2 + 4y - 8 + z - 4 = 0$$

$$2x + 4y + z - 14 = 0$$

$$\boxed{2x + 4y + z = 14}$$

**Step 5:** Find the parametric equations of the normal line.

The line passes through  $(1,2,4)$  with direction vector  $\langle 2,4,1 \rangle$ .

$$\begin{aligned}x &= 1 + 2t \\ y &= 2 + 4t \\ z &= 4 + 1t\end{aligned}$$

**Final Answer for Normal Line:**

- **Tangent Plane:**  $2x + 4y + z - 14 = 0$
- **Normal Line:**  $x = 1 + 2t, y = 2 + 4t, z = 4 + t$

**Problem 9 (Normal to a sphere).** Find the equations of the normal line to the sphere  $x^2 + y^2 + z^2 = 6$  at  $(a, b, c)$  and show it passes through the origin

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**Answer:**

**Concept:** For a sphere  $F(x, y, z) = x^2 + y^2 + z^2 - 6 = 0$ , the normal at any point is given by  $\nabla F$ .

**Step 1:** Find  $\nabla F$ .

$$F_x = 2x, F_y = 2y, F_z = 2z$$

So,  $\nabla F = \langle 2x, 2y, 2z \rangle$  At point  $(a, b, c)$ , this is  $\langle 2a, 2b, 2c \rangle$ .

A simpler normal vector is  $\vec{n} = \langle a, b, c \rangle$  (scaling by 2 doesn't change direction).

**Step 2:** Write the equation of the normal line.

It passes through  $(a, b, c)$  and is parallel to  $\vec{n} = \langle a, b, c \rangle$ .

Parametric equations:

$$x = a + at = a(1 + t)$$

$$y = b + bt = b(1 + t)$$

$$z = c + ct = c(1 + t)$$

**Step 3:** Show it passes through the origin  $(0, 0, 0)$ .

We need to find a parameter  $t$  such that  $x=0, y=0, z=0$ .

$$a(1 + t) = 0$$

$$b(1 + t) = 0$$

$$c(1 + t) = 0$$

Since  $(a, b, c)$  is on the sphere ( $a^2 + b^2 + c^2 = 6$ ), it cannot be the origin itself. Therefore,  $a, b, c$  are not all zero. The only solution to all three equations is  $1 + t = 0$ , i.e.,  $t = -1$ .

Substituting  $t = -1$  into the parametric equations:

$$x = a(1 + (-1)) = a(0) = 0$$

$$y = b(0) = 0$$

$$z = c(0) = 0$$

This proves the line passes through the origin.

**Final Answer for Normal Line:**

$$x = a(1 + t)$$

$$y = b(1 + t)$$

$$z = c(1 + t)$$

The line passes through the origin when  $t = -1$ .

**Problem 10 (Distance to quadratic surface). Find the minimum distance from the surface  $x^2 + y^2 - z^2 = 1$  to the origin using Lagrange multipliers.**

**Answer:**

**Concept:** We want to minimize the distance  $d = \sqrt{x^2 + y^2 + z^2}$  from a point  $(x, y, z)$  on the surface to the origin  $(0, 0, 0)$ . To simplify calculations, we minimize the *square* of the distance  $f(x, y, z) = x^2 + y^2 + z^2$  subject to the constraint  $g(x, y, z) = x^2 + y^2 - z^2 - 1 = 0$ .

**Step 1:** Set up the Lagrange multiplier equations:  $\nabla f = \lambda \nabla g$ .

$$\nabla f = \langle 2x, 2y, 2z \rangle$$

$$\nabla g = \langle 2x, 2y, -2z \rangle$$

This gives the system:

$$\begin{cases} 2x = \lambda(2x) & (1) \\ 2y = \lambda(2y) & (2) \\ 2z = \lambda(-2z) & (3) \\ x^2 + y^2 - z^2 = 1 & (4) \end{cases}$$

**Step 2:** Solve the system. Notice equations (1) and (2) are similar.

$$\text{From (1): } 2x - 2\lambda x = 0 \implies 2x(1 - \lambda) = 0 \implies \text{either } x = 0 \text{ or } \lambda = 1.$$

$$\text{From (2): } 2y - 2\lambda y = 0 \implies 2y(1 - \lambda) = 0 \implies \text{either } y = 0 \text{ or } \lambda = 1.$$

$$\text{From (3): } 2z + 2\lambda z = 0 \implies 2z(1 + \lambda) = 0 \implies \text{either } z = 0 \text{ or } \lambda = -1.$$

We must consider cases based on the values of  $\lambda$ .

- **Case 1:**  $\lambda=1$ . Substitute into equation (3):

$$2z(1 + 1) = 0 \implies 4z = 0 \implies z = 0.$$

Now substitute  $z=0$  and  $\lambda=1$  into constraint (4):

$$x^2 + y^2 - (0)^2 = 1 \implies x^2 + y^2 = 1.$$

On this circle, the function we are minimizing is  $f(x, y, 0) = x^2 + y^2 = 1$ . So the squared distance is 1.

- **Case 2:**  $\lambda=-1$ . Substitute into equation (1):

$$2x(1 - (-1)) = 0 \implies 2x(2) = 0 \implies x = 0.$$

Substitute into equation (2):

$$2y(1 - (-1)) = 0 \implies 2y(2) = 0 \implies y = 0.$$

Now substitute  $x=0, y=0, \lambda=-1$  into constraint (4):

$$(0)^2 + (0)^2 - z^2 = 1 \implies -z^2 = 1 \implies z^2 = -1.$$

This has no real solutions. So this case is invalid.

- **Case 3:**  $\lambda$  is neither 1 nor -1. Then from (1), (2), and (3), we must have  $x=0, y=0, z=0$ . Substituting into constraint (4):  $0+0-0=1 \implies 0=1$ , which is false. This case is also invalid.

**Step 3:** The only valid case is Case 1, which gives a squared distance of 1. Therefore, the minimum distance from the surface to the origin is  $\sqrt{1} = 1$ .

**Final Answer:** The minimum distance is 1.

**Problem 11 (Box in first octant under a plane).** *A rectangular box lies in the first octant with one corner at the origin and the opposite corner on the plane  $x + 2y + 3z = 3$ . Find the maximum possible volume.*

**Answer:**

**Step 1:** Set up the optimization problem using Lagrange Multipliers.

We want to maximize  $V = xyz$  subject to  $g(x, y, z) = x + 2y + 3z - 3 = 0$ .

The Lagrange equation is  $\nabla V = \lambda \nabla g$ .

**Step 2:** Calculate the gradients.

$$\nabla V = \left\langle \frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z} \right\rangle = \langle yz, xz, xy \rangle$$

$$\nabla g = \langle 1, 2, 3 \rangle$$

This gives us the system of equations:

$$\begin{cases} yz = \lambda \cdot 1 & (1) \\ xz = \lambda \cdot 2 & (2) \\ xy = \lambda \cdot 3 & (3) \\ x + 2y + 3z = 3 & (4) \end{cases}$$

**Step 3:** Solve for  $x, y, z$  in terms of  $\lambda$ . Assuming  $x, y, z > 0$  (to be in the first octant and have positive volume):

$$\text{From (1): } \lambda = yz$$

$$\text{From (2): } \lambda = \frac{xz}{2}$$

$$\text{From (3): } \lambda = \frac{xy}{3}$$

Set the expressions for  $\lambda$  equal to each other.

$$\text{From (1) = (2): } yz = \frac{xz}{2}. \text{ Since } z > 0, \text{ divide both sides by } z: y = \frac{x}{2} \text{ or } x = 2y.$$

$$\text{From (1) = (3): } yz = \frac{xy}{3}. \text{ Since } y > 0, \text{ divide both sides by } y: z = \frac{x}{3}.$$

**Step 4:** Substitute  $x=2y$  and  $z=x/3=2y/3$  into the constraint equation (4).

$$x + 2y + 3z = 3$$

$$(2y) + 2y + 3\left(\frac{2y}{3}\right) = 3$$

$$2y + 2y + 2y = 3$$

$$6y = 3$$

$$y = \frac{1}{2}$$

**Step 5:** Find  $x$  and  $z$ .

$$x = 2y = 2 \cdot \frac{1}{2} = 1$$

$$z = \frac{2y}{3} = \frac{2 \cdot \frac{1}{2}}{3} = \frac{1}{3}$$

The corner on the plane is at  $(1, 1/2, 1/3)$ .

**Step 6:** Calculate the maximum volume.

$$V = xyz = (1) \cdot \left(\frac{1}{2}\right) \cdot \left(\frac{1}{3}\right) = \frac{1}{6}$$

**Final Answer:** The maximum possible volume is  $1/6$ .

**Problem 12.** Extremize  $x^2 + y^2 + z^2 + xy + yz + zx$  subject to  $x + y + z = 1$  and  $x + 2y + 3z = 1$ .

**Answer:**

**Concept:** This is a constrained optimization problem with two constraints. We use two Lagrange multipliers,  $\lambda$  and  $\mu$ . The equation is  $\nabla f = \lambda \nabla g_1 + \mu \nabla g_2$ , where:

$$g_1(x, y, z) = x + y + z - 1 = 0$$

$$g_2(x, y, z) = x + 2y + 3z - 1 = 0$$

**Step 1:** Compute all gradients.

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \langle 2x + y + z, 2y + x + z, 2z + x + y \rangle$$

$$\nabla g_1 = \langle 1, 1, 1 \rangle$$

$$\nabla g_2 = \langle 1, 2, 3 \rangle$$

This gives the vector equation:

$$\langle 2x + y + z, 2y + x + z, 2z + x + y \rangle = \lambda \langle 1, 1, 1 \rangle + \mu \langle 1, 2, 3 \rangle$$

This breaks down into a system of 5 equations (3 from gradients, 2 constraints):

$$\begin{cases} 2x + y + z = \lambda + \mu & (1) \\ x + 2y + z = \lambda + 2\mu & (2) \\ x + y + 2z = \lambda + 3\mu & (3) \\ x + y + z = 1 & (4) \\ x + 2y + 3z = 1 & (5) \end{cases}$$

**Step 2:** Notice the symmetry. Subtract equation (1) from equation (2):

$$(x + 2y + z) - (2x + y + z) = (\lambda + 2\mu) - (\lambda + \mu)$$

$$-x + y = \mu$$

$$\text{So, } y = x + \mu \quad (6)$$

Subtract equation (2) from equation (3):

$$(x + y + 2z) - (x + 2y + z) = (\lambda + 3\mu) - (\lambda + 2\mu)$$

$$-y + z = \mu$$

$$\text{So, } z = y + \mu \quad (7)$$

**Step 3:** Substitute (6) and (7) into the constraint (4):  $x + y + z = 1$

$$x + (x + \mu) + ((x + \mu) + \mu) = 1$$

$$3x + 3\mu = 1$$

$$x + \mu = \frac{1}{3} \quad (8)$$

From (6),  $y = x + \mu$ , so equation (8) gives  $y = 1/3$ .

**Step 4:** Substitute  $y = 1/3$  into constraint (5):  $x + 2y + 3z = 1$

$$x + 2\left(\frac{1}{3}\right) + 3z = 1$$

$$x + \frac{2}{3} + 3z = 1$$

$$x + 3z = \frac{1}{3} \quad (9)$$

**Step 5:** We also have constraint (4):  $x+y+z=1$

$$x + \frac{1}{3} + z = 1$$

$$x + z = \frac{2}{3} \quad (10)$$

**Step 6:** Solve the system of equations (9) and (10).

Equation (9):  $x+3z=1/3$

Equation (10):  $x+z=2/3$

Subtract (10) from (9):  $(x + 3z) - (x + z) = \frac{1}{3} - \frac{2}{3}$

$$2z = -\frac{1}{3}$$

$$z = -\frac{1}{6}$$

Substitute  $z = -1/6$  into (10):

$$x + \left(-\frac{1}{6}\right) = \frac{2}{3}$$

$$x = \frac{2}{3} + \frac{1}{6} = \frac{4}{6} + \frac{1}{6} = \frac{5}{6}$$

**Step 7:** We have  $x=5/6, y=1/3, z=-1/6$ . This satisfies both constraints (4) and (5). This is the only critical point.

**Step 8:** Find the value of the function at this point.

$$f\left(\frac{5}{6}, \frac{1}{3}, -\frac{1}{6}\right) = \left(\frac{5}{6}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(-\frac{1}{6}\right)^2 + \left(\frac{5}{6}\right)\left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)\left(-\frac{1}{6}\right) + \left(\frac{5}{6}\right)\left(-\frac{1}{6}\right)$$

Calculate each term:

$$= \frac{25}{36} + \frac{1}{9} + \frac{1}{36} + \frac{5}{18} - \frac{1}{18} - \frac{5}{36}$$

Convert all terms to have a denominator of 36:

$$\begin{aligned} &= \frac{25}{36} + \frac{4}{36} + \frac{1}{36} + \frac{10}{36} - \frac{2}{36} - \frac{5}{36} \\ &= \frac{(25+4+1+10-2-5)}{36} = \frac{33}{36} = \frac{11}{12} \end{aligned}$$

**Interpretation:** With only one critical point and the function being quadratic (an ellipsoid), this point represents an extremum. Given the coefficients are positive, it is likely a minimum.

**Final Answer:** The function has an extremum value of  $\frac{11}{12}$  at the point  $\left(\frac{5}{6}, \frac{1}{3}, -\frac{1}{6}\right)$

**Problem 13 (Unconstrained).** Find and classify the critical points of  $f(x, y) = x^3 + y^3 - 3x - 12y + 20$ .

---

**Answer:**

**Step 1:** Find the first partial derivatives and set them to zero.

$$f_x = 3x^2 - 3 = 0 \implies 3(x^2 - 1) = 0 \implies x = 1, x = -1$$

$$f_y = 3y^2 - 12 = 0 \implies 3(y^2 - 4) = 0 \implies y = 2, y = -2$$

**Step 2:** The critical points are all combinations of these x and y values:

(1,2),(1,-2),(-1,2),(-1,-2)

**Step 3:** Classify each using the Second Derivative Test. Find the second partials.

$$f_{xx} = 6x$$

$$f_{yy} = 6y$$

$$f_{xy} = 0$$

The discriminant is  $D = f_{xx}f_{yy} - (f_{xy})^2 = (6x)(6y) - 0 = 36xy$ .

Evaluate  $D$  and  $f_{xx}$  at each point:

1. At (1, 2):  $D = 36(1)(2) = 72 > 0$ ,  $f_{xx} = 6(1) = 6 > 0$ . **Local Minimum.**
2. At (1, -2):  $D = 36(1)(-2) = -72 < 0$ . **Saddle Point.**
3. At (-1, 2):  $D = 36(-1)(2) = -72 < 0$ . **Saddle Point.**
4. At (-1, -2):  $D = 36(-1)(-2) = 72 > 0$ ,  $f_{xx} = 6(-1) = -6 < 0$ . **Local Maximum.**

**Final Answer:**

- (1,2) is a **Local Minimum.**
- (1,-2) is a **Saddle Point.**
- (-1,2) is a **Saddle Point.**
- (-1,-2) is a **Local Maximum.**

**Problem 14 (Unconstrained).** Determine local maxima, minima, and saddle points of  $f(x,y) = x^4 + y^4 - 2x^2 + 4y$ .

---

**Answer:**

**Step 1:** Find the critical points.

$$f_x = 4x^3 - 4x = 4x(x^2 - 1) = 4x(x - 1)(x + 1) = 0 \implies x = 0, x = 1, x = -1$$

$$f_y = 4y^3 + 4 = 4(y^3 + 1) = 0 \implies y^3 = -1 \implies y = -1$$

The critical points are: (0,-1),(1,-1),(-1,-1).

**Step 2:** Find the second partial derivatives for the D-test.

$$f_{xx} = 12x^2 - 4$$

$$f_{yy} = 12y^2$$

$$f_{xy} = 0$$

$$\text{Discriminant: } D = (12x^2 - 4)(12y^2) - (0)^2 = 12y^2(12x^2 - 4)$$

**Step 3:** Classify each point.

1. At  $(0, -1)$ :

$$f_{xx} = 12(0)^2 - 4 = -4$$

$$D = 12(-1)^2 \cdot (12(0)^2 - 4) = 12 \cdot 1 \cdot (-4) = -48 < 0$$

**Saddle Point.**

2. At  $(1, -1)$ :

$$f_{xx} = 12(1)^2 - 4 = 8 > 0$$

$$D = 12(-1)^2 \cdot (12(1)^2 - 4) = 12 \cdot 1 \cdot 8 = 96 > 0$$

**Local Minimum.**

3. At  $(-1, -1)$ :

$$f_{xx} = 12(-1)^2 - 4 = 8 > 0$$

$$D = 12(-1)^2 \cdot (12(-1)^2 - 4) = 12 \cdot 1 \cdot 8 = 96 > 0$$

**Local Minimum.**

**Final Answer:**

- $(0, -1)$  is a **Saddle Point**.
- $(1, -1)$  is a **Local Minimum**.
- $(-1, -1)$  is a **Local Minimum**.

**Problem 15 (Constrained: line).** Use Lagrange multipliers to find extrema of  $f(x, y) = x^2 + y^2$  subject to  $x + y = 1$ .

---

**Answer:**

This problem is identical to **Problem 5**. The solution is the same.

**Final Answer:** The function  $f$  has a **minimum** value of  $1/2$  at the point  $(1/2, 1/2)$ .

**Problem 16 (Constrained: quadratic).** Find the maximum and minimum of  $f(x, y) = x^2 + y^2$  subject to  $x^2 - 2x + y^2 - 4y = 0$ .

---

**Answer:**

**Step 1: Define the Constraint Function**

First, we define the constraint in the form  $g(x, y) = 0$ :

$$g(x, y) = x^2 - 2x + y^2 - 4y = 0$$

**Step 2: Set Up the Lagrange Multiplier Equations**

We want to optimize  $f(x, y) = x^2 + y^2$  subject to  $g(x, y) = 0$ .

The necessary condition is  $\nabla f = \lambda \nabla g$ .

Calculate the gradients:

$$\nabla f = \langle 2x, 2y \rangle$$

$$\nabla g = \langle 2x - 2, 2y - 4 \rangle$$

This gives us the system of equations:

$$\begin{cases} 2x = \lambda(2x - 2) & (1) \\ 2y = \lambda(2y - 4) & (2) \\ x^2 - 2x + y^2 - 4y = 0 & (3) \end{cases}$$

**Step 3: Solve the System of Equations from (1) and (2)**

Let's simplify and solve equations (1) and (2) for potential relationships between  $x, y$ , and  $\lambda$ .

**From Equation (1):**

$$2x = 2\lambda x - 2\lambda$$

Bring all terms to one side:

$$2x - 2\lambda x + 2\lambda = 0$$

Factor:

$$2x(1 - \lambda) + 2\lambda = 0$$

Divide both sides by 2:

$$x(1 - \lambda) + \lambda = 0$$

$$x(1 - \lambda) = -\lambda$$

If  $\lambda \neq 1$ , we can solve for  $x$ :

$$x = \frac{-\lambda}{1 - \lambda} = \frac{\lambda}{\lambda - 1}$$

**From Equation (2):**

$$2y = 2\lambda y - 4\lambda$$

Bring all terms to one side:

$$2y - 2\lambda y + 4\lambda = 0$$

Factor:

$$2y(1 - \lambda) + 4\lambda = 0$$

Divide both sides by 2:

$$y(1 - \lambda) + 2\lambda = 0$$

$$y(1 - \lambda) = -2\lambda$$

If  $\lambda \neq 1$ , we can solve for  $y$ :

$$y = \frac{-2\lambda}{1 - \lambda} = \frac{2\lambda}{\lambda - 1}$$

**The Case  $\lambda=1$ :**

Substitute  $\lambda=1$  into the original equations (1) and (2):

Equation (1):  $2x=1 \cdot (2x-2) \Rightarrow 2x=2x-2 \Rightarrow 0=-2$ . This is a contradiction.

Equation (2):  $2y=1 \cdot (2y-4) \Rightarrow 2y=2y-4 \Rightarrow 0=-4$ . Also a contradiction.

Therefore,  $\lambda=1$  is **not a valid solution**. Our expressions for  $x$  and  $y$  in terms of  $\lambda$  are valid for all relevant  $\lambda$ .

**Step 4: Substitute  $x$  and  $y$  into the Constraint (3)**

We have:

$$x = \frac{\lambda}{\lambda - 1}$$

$$y = \frac{2\lambda}{\lambda - 1}$$

Let's substitute these into the constraint equation (3):  $x^2 - 2x + y^2 - 4y = 0$ .

First, compute each term:

$$1. x^2 = \left(\frac{\lambda}{\lambda-1}\right)^2 = \frac{\lambda^2}{(\lambda-1)^2}$$

$$2. -2x = -2 \cdot \frac{\lambda}{\lambda-1} = \frac{-2\lambda}{\lambda-1}$$

$$3. y^2 = \left(\frac{2\lambda}{\lambda-1}\right)^2 = \frac{4\lambda^2}{(\lambda-1)^2}$$

$$4. -4y = -4 \cdot \frac{2\lambda}{\lambda-1} = \frac{-8\lambda}{\lambda-1}$$

Now, add them all together. To combine them, get a common denominator, which is  $(\lambda-1)^2$ :

$$\begin{aligned} x^2 - 2x + y^2 - 4y &= \frac{\lambda^2}{(\lambda-1)^2} + \frac{-2\lambda}{\lambda-1} + \frac{4\lambda^2}{(\lambda-1)^2} + \frac{-8\lambda}{\lambda-1} \\ &= \frac{\lambda^2 + 4\lambda^2}{(\lambda-1)^2} + \frac{-2\lambda - 8\lambda}{\lambda-1} \\ &= \frac{5\lambda^2}{(\lambda-1)^2} + \frac{-10\lambda}{\lambda-1} \end{aligned}$$

Now, express the second term with the common denominator  $(\lambda-1)^2$ :

$$\frac{-10\lambda}{\lambda-1} = \frac{-10\lambda(\lambda-1)}{(\lambda-1)^2}$$

So the entire expression becomes:

$$\frac{5\lambda^2}{(\lambda-1)^2} + \frac{-10\lambda(\lambda-1)}{(\lambda-1)^2} = \frac{5\lambda^2 - 10\lambda(\lambda-1)}{(\lambda-1)^2}$$

Simplify the numerator:

$$5\lambda^2 - 10\lambda(\lambda-1) = 5\lambda^2 - 10\lambda^2 + 10\lambda = -5\lambda^2 + 10\lambda$$

Therefore, the constraint equation becomes:

$$\frac{-5\lambda^2 + 10\lambda}{(\lambda-1)^2} = 0$$

A fraction is zero only if its numerator is zero (and the denominator is not zero, which it isn't since  $\lambda \neq 1$ ):

$$-5\lambda^2 + 10\lambda = 0$$

$$-5\lambda(\lambda - 2) = 0$$

This gives two solutions:

$$\lambda=0 \text{ or } \lambda=2$$

### Step 5: Find the Corresponding Points (x,y) and Function Values f(x,y)

Case 1:  $\lambda=0$

$$x = \frac{0}{0-1} = 0$$

$$y = \frac{2 \cdot 0}{0-1} = 0$$

Check if (0,0) satisfies the constraint (3):

$$(0)^2 - 2(0) + (0)^2 - 4(0) = 0. \text{ Yes, it does.}$$

$$f(0,0) = 0^2 + 0^2 = 0$$

Case 2:  $\lambda=2$

$$x = \frac{2}{2-1} = \frac{2}{1} = 2$$

$$y = \frac{2 \cdot 2}{2-1} = \frac{4}{1} = 4$$

Check if (2,4) satisfies the constraint (3):

$$(2)^2 - 2(2) + (4)^2 - 4(4) = 4 - 4 + 16 - 16 = 0. \text{ Yes, it does.}$$

$$f(2, 4) = (2)^2 + (4)^2 = 4 + 16 = 20$$

### Step 6: Final Answer

We have found two critical points on the constraint curve. The function  $f(x,y)=x^2+y^2$  represents the squared distance from the origin. The point (0,0) is clearly the minimum. The point (2,4) gives a larger value. There are no other critical points, so these must be the absolute extrema for this continuous function on this closed and bounded curve (a circle).

### Final Answer:

- The minimum value of  $f$  is  $0$ , attained at  $(0, 0)$ .
- The maximum value of  $f$  is  $20$ , attained at  $(2, 4)$ .

**Problem 17 (Distance).** Find the minimum distance from the surface  $x^2 + y^2 - z^2 = 1$  to the origin.

---

**Answer:**

This problem is identical to **Problem 10**. The solution is the same.

**Final Answer:** The minimum distance is 11.

**Problem 18 (Inscribed box).** Find the volume of the largest axis-aligned rectangular box that can be inscribed in the ellipsoid  $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ .

---

**Answer:**

**Step 1:** Set up the optimization problem.

Maximize  $V=8xyz$  subject to  $g(x,y,z)=0$ .

It is easier to maximize  $F=\ln(V/8)=\ln(xyz)=\ln x+\ln y+\ln z$ , as the maximum of  $F$  occurs at the same point as the maximum of  $V$ .

**Step 2:** Use Lagrange Multipliers:  $\nabla F=\lambda\nabla g$ .

$$\nabla F = \left\langle \frac{1}{x}, \frac{1}{y}, \frac{1}{z} \right\rangle$$

$$\nabla g = \left\langle \frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2} \right\rangle$$

This gives the system:

$$\begin{cases} \frac{1}{x} = \lambda \frac{2x}{a^2} & (1) \\ \frac{1}{y} = \lambda \frac{2y}{b^2} & (2) \\ \frac{1}{z} = \lambda \frac{2z}{c^2} & (3) \\ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 & (4) \end{cases}$$

**Step 3:** Solve for  $\lambda$  from each equation.

$$\text{From (1): } \lambda = \frac{a^2}{2x^2}$$

$$\text{From (2): } \lambda = \frac{b^2}{2y^2}$$

$$\text{From (3): } \lambda = \frac{c^2}{2z^2}$$

$$\text{Set them equal: } \frac{a^2}{2x^2} = \frac{b^2}{2y^2} = \frac{c^2}{2z^2}$$

This implies  $\frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2}$ . Let this common ratio be  $k$ .

$$\text{So, } x^2 = ka^2, y^2 = kb^2, z^2 = kc^2.$$

**Step 4:** Substitute into the constraint (4).

$$\frac{ka^2}{a^2} + \frac{kb^2}{b^2} + \frac{kc^2}{c^2} = 1$$

$$k + k + k = 1$$

$$3k = 1$$

$$k = \frac{1}{3}$$

**Step 5:** Find the dimensions of the optimal box.

$$x = \frac{a}{\sqrt{3}}, y = \frac{b}{\sqrt{3}}, z = \frac{c}{\sqrt{3}}$$

$$\text{The full dimensions are } 2x = \frac{2a}{\sqrt{3}}, 2y = \frac{2b}{\sqrt{3}}, 2z = \frac{2c}{\sqrt{3}}.$$

**Step 6:** Calculate the maximum volume.

$$V_{\max} = 8xyz = 8 \cdot \frac{a}{\sqrt{3}} \cdot \frac{b}{\sqrt{3}} \cdot \frac{c}{\sqrt{3}} = \frac{8abc}{3\sqrt{3}}$$

**Final Answer:** The maximum volume is  $\boxed{\frac{8abc}{3\sqrt{3}}}$ .

**Problem 19 (Three-part split).** Divide 24 into three parts  $a, b, c$  so that the product  $a, b^2, c^3$  is maximized. Find the maximum value.

---

**Answer:**

**Step 1:** Define the variables and the constraint.

The sum is fixed:  $a+b+c=24$ .

We want to maximize  $P=a \cdot b^2 \cdot c^3$

**Step 2:** Use Lagrange Multipliers. Let  $F = \ln P = \ln a + 2 \ln b + 3 \ln c$ .

The constraint is  $g(a, b, c) = a + b + c - 24 = 0$ .

**Step 3:** Set up the Lagrange equation  $\nabla F = \lambda \nabla g$ .

$$\begin{aligned}\nabla F &= \left\langle \frac{1}{a}, \frac{2}{b}, \frac{3}{c} \right\rangle \\ \nabla g &= \langle 1, 1, 1 \rangle\end{aligned}$$

This gives the system:

$$\begin{cases} \frac{1}{a} = \lambda & (1) \\ \frac{2}{b} = \lambda & (2) \\ \frac{3}{c} = \lambda & (3) \\ a + b + c = 24 & (4) \end{cases}$$

**Step 4:** Solve for a,b,c in terms of  $\lambda$ .

$$\begin{aligned}\text{From (1): } a &= \frac{1}{\lambda} \\ \text{From (2): } b &= \frac{2}{\lambda} \\ \text{From (3): } c &= \frac{3}{\lambda}\end{aligned}$$

**Step 5:** Substitute into the constraint (4).

$$\begin{aligned}\frac{1}{\lambda} + \frac{2}{\lambda} + \frac{3}{\lambda} &= 24 \\ \frac{6}{\lambda} &= 24 \\ \lambda &= \frac{6}{24} = \frac{1}{4}\end{aligned}$$

**Step 6:** Find a,b,c.

$$\begin{aligned}a &= \frac{1}{(1/4)} = 4 \\ b &= \frac{2}{(1/4)} = 8 \\ c &= \frac{3}{(1/4)} = 12\end{aligned}$$

**Step 7:** Calculate the maximum product.

$$P_{\max} = a \cdot b^2 \cdot c^3 = (4) \cdot (8)^2 \cdot (12)^3 = 4 \cdot 64 \cdot 1728$$

Calculate stepwise:

$$4 \times 64 = 256$$

$$\begin{aligned}256 \times 1728 &= 256 \times (1700 + 28) = (256 \times 1700) + (256 \times 28) = 435,200 + \\ &7,168 = 442,368\end{aligned}$$

**Final Answer:** The maximum value is 442368, achieved when  $a = 4, b = 8, c = 12$ .

**Problem 20 (Quadratic form on disk).** Find the maximum and minimum of  $f(x,y) = 4x^2 + 10y^2$  on the disk  $x^2 + y^2 \leq 4$ .

---

**Answer:**

**Step 1: Find Critical Points Inside the Disk**

Find where  $\nabla f = \langle 0, 0 \rangle$ .

$$f_x = 8x = 0 \implies x = 0$$

$$f_y = 20y = 0 \implies y = 0$$

The point (0,0) is inside the disk.  $f(0,0)=0$ . This is a candidate for the minimum.

**Step 2: Find Extrema on the Boundary**

The boundary is the circle  $x^2+y^2=4$ . Use Lagrange Multipliers with constraint

$$g(x, y) = x^2 + y^2 - 4 = 0.$$

$$\nabla f = \langle 8x, 20y \rangle$$

$$\nabla g = \langle 2x, 2y \rangle$$

Lagrange:  $\langle 8x, 20y \rangle = \lambda \langle 2x, 2y \rangle$

This gives:

$$\begin{cases} 8x = \lambda(2x) & (1) \\ 20y = \lambda(2y) & (2) \\ x^2 + y^2 = 4 & (3) \end{cases}$$

**Step 3: Solve the System**

- If  $x \neq 0$ , from (1):  $8 = 2\lambda \implies \lambda = 4$ .  
Substitute  $\lambda = 4$  into (2):  $20y = 8y \implies 12y = 0 \implies y = 0$ .  
Substitute  $y = 0$  into (3):  $x^2 = 4 \implies x = \pm 2$ .  
Candidates:  $(2, 0)$  and  $(-2, 0)$ .  $f(\pm 2, 0) = 4(4) + 10(0) = 16$ .
- If  $y \neq 0$ , from (2):  $20 = 2\lambda \implies \lambda = 10$ .  
Substitute  $\lambda = 10$  into (1):  $8x = 20x \implies -12x = 0 \implies x = 0$ .  
Substitute  $x = 0$  into (3):  $y^2 = 4 \implies y = \pm 2$ .  
Candidates:  $(0, 2)$  and  $(0, -2)$ .  $f(0, \pm 2) = 4(0) + 10(4) = 40$ .
- The case  $x = 0, y = 0$  is already considered and is not on the boundary.

- **Step 4: Evaluate Candidates**

- Interior:  $f(0,0)=0$
- Boundary:  $f(\pm 2,0)=16, f(0,\pm 2)=40$

**Conclusion:**

- **Absolute Minimum:**  $\boxed{0}$  at  $\boxed{(0, 0)}$
- **Absolute Maximum:**  $\boxed{40}$  at  $\boxed{(0, 2)}$  and  $\boxed{(0, -2)}$

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