Subject Name & Code:

MATHEMATICS I- BE01R00041

TWA – 6

Topic: (Sequences and Basic Convergence Methods)

Q-1: Convergence of Sequences

Answer:

(a) $a_n = \frac{5n-2}{2n+7}$

1. Identify the highest power of n in the denominator: n.
2. Divide both the numerator and the denominator by n:

$$a_n = \frac{5 - \frac{2}{n}}{2 + \frac{7}{n}}$$

3. Take the limit as $n \to \infty$:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{5 - \frac{2}{n}}{2 + \frac{7}{n}} = \frac{5-0}{2+0} = \frac{5}{2}$$

4. **Conclusion:** The sequence converges. Its limit is $\frac{5}{2}$.

(b) $a_n = (1 + \frac{3}{n})^{1/n}$

1. Let $L = \lim_{n\to\infty} a_n$.
2. Take the natural logarithm of both sides:

$$\ln\ L = \lim_{n\to\infty} \ln\left((1+\frac{3}{n})^{1/n}\right) = \lim_{n\to\infty} \frac{1}{n} \cdot \ln\left(1+\frac{3}{n}\right)$$

3. The limit is of the form $0 \cdot 0$. Rewrite it:

$$\ln\ L = \lim_{n\to\infty} \frac{\ln(1+\frac{3}{n})}{n}$$

As $n \to \infty$, the numerator $\ln\left(1+\frac{3}{n}\right) \to \ln(1) = 0$ and the denominator $n \to \infty$. This is a $\frac{0}{\infty}$ form, which evaluates to 0.

4. Therefore, $\ln L = 0$.
5. Exponentiate both sides to solve for L: $L = e^0 = 1$.
6. **Conclusion:** The sequence converges. Its limit is 1.

(c) $a_n = \frac{(-1)^n}{\sqrt{n}+1}$

1. Consider the absolute value of the sequence:

$$|a_n| = \left|\frac{(-1)^n}{\sqrt{n}+1}\right| = \frac{1}{\sqrt{n}+1}$$

2. Observe that $\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{1}{\sqrt{n}+1} = 0$.
3. If the absolute value of a sequence converges to 0, the sequence itself also converges to 0.
4. **Conclusion:** The sequence converges. Its limit is 0.

(d) $a_n = \sin\left(\frac{1}{n^2}\right)$

1. As $n \to \infty$, the argument $\frac{1}{n^2} \to 0$.
2. We use the known limit: $\lim_{x\to 0} \frac{\sin x}{x} = 1$, which implies $\sin x \approx x$ for very small x.
3. Therefore, $\sin\left(\frac{1}{n^2}\right) \approx \frac{1}{n^2}$ for large n.
4. More formally, since $\lim_{x\to 0} \sin x = 0$, we have:

$$\lim_{n\to\infty} \sin\left(\frac{1}{n^2}\right) = \sin(0) = 0$$

5. **Conclusion:** The sequence converges. Its limit is 0.

Q2. Squeeze Theorem Problems

Answer:

The Squeeze Theorem states that if $b_n \leq a_n \leq c_n$ for all $n > N$ and $\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} a_n = L$.

(a) $a_n = n\sin\left(\frac{1}{n^3}\right)$

1. Recall the inequality $-1 \leq \sin x \leq 1$. A more useful inequality for small x is $|\sin x| \leq |x|$.
2. Apply this inequality:

$$|a_n| = \left|n\sin\left(\frac{1}{n^3}\right)\right| \leq n \cdot \left|\frac{1}{n^3}\right| = \frac{1}{n^2}$$

3. This gives us:

$$-\frac{1}{n^2} \leq n\sin\left(\frac{1}{n^3}\right) \leq \frac{1}{n^2}$$

4. Now, $\lim_{n\to\infty}\left(-\frac{1}{n^2}\right) = 0$ and $\lim_{n\to\infty} \frac{1}{n^2} = 0$.
5. **Conclusion:** By the Squeeze Theorem, $\lim_{n\to\infty} n\sin\left(\frac{1}{n^3}\right) = 0$.

(b) $a_n = \frac{\tan(1/n)}{1+2n}$

1. For $n \geq 1$, $1/n \in (0,1] \subset (-\pi/2, \pi/2)$, where $\tan x$ is positive for $x > 0$.
2. We know that for $x > 0$, $\tan x > x$.
3. Also, since $1 + 2n > 0$, we can write:

$$0 \leq \frac{\tan(1/n)}{1+2n} \leq \frac{\tan(1/n)}{2n}$$

4. Using the inequality $\tan x > x$, we get a looser but sufficient bound:

$$0 \leq a_n \leq \frac{\tan(1/n)}{2n} \leq \frac{1/n}{2n} = \frac{1}{2n^2}$$

(The step $\tan(1/n) \leq 1/n$ is not generally true for $n = 1$, but for large n, $1/n$ is small and $\tan(1/n) \approx 1/n$. A more rigorous bound is $|\tan x| \leq 2|x|$ for $|x| < \pi/4$, which holds for $n > 1$).

5. A simpler, correct approach: $|\tan(1/n)| \leq 2 \cdot |1/n|$ for large n.

$$|a_n| = \left|\frac{\tan(1/n)}{1+2n}\right| \leq \frac{2\cdot(1/n)}{2n} = \frac{1}{n^2}$$

So, $-\frac{1}{n^2} \leq a_n \leq \frac{1}{n^2}$.

6. Since $\lim_{n\to\infty} \pm\frac{1}{n^2} = 0$.
7. **Conclusion:** By the Squeeze Theorem, $\lim_{n\to\infty}\frac{\tan(1/n)}{1+2n} = 0$.

(c) $a_n = \frac{\cos(n)}{n^2}$

1. Use the fact that $-1 \leq \cos(n) \leq 1$.
2. Divide the entire inequality by n^2 (which is always positive):

$$-\frac{1}{n^2} \leq \frac{\cos(n)}{n^2} \leq \frac{1}{n^2}$$

3. Now, $\lim_{n\to\infty}(-\frac{1}{n^2}) = 0$ and $\lim_{n\to\infty}\frac{1}{n^2} = 0$.
4. **Conclusion:** By the Squeeze Theorem, $\lim_{n\to\infty}\frac{\cos(n)}{n^2} = 0$.

Q3. Limits by Continuous Function Theorem

Answer:

Let $a_n = (3n^2+1)^{1/n}$.

(a) Compute $\lim a_n$**.**

1. Let $L = \lim_{n\to\infty}(3n^2+1)^{1/n}$.
2. Take the natural logarithm of both sides:

$$\ln L = \lim_{n\to\infty} \ln\left((3n^2+1)^{1/n}\right) = \lim_{n\to\infty}\frac{\ln(3n^2+1)}{n}$$

3. As $n\to\infty$, both numerator and denominator $\to\infty$. We can apply L'Hôpital's Rule.
4. Differentiate numerator and denominator with respect to n:

- Derivative of numerator: $\frac{d}{dn}[\ln{(3n^2+1)}] = \frac{6n}{3n^2+1}$
- Derivative of denominator: $\frac{d}{dn}[n] = 1$

5. Apply L'Hôpital's Rule:

$$\ln\, L = \lim_{n\to\infty} \frac{\frac{6n}{3n^2+1}}{1} = \lim_{n\to\infty} \frac{6n}{3n^2+1}$$

6. Divide numerator and denominator by n^2:

$$\ln\, L = \lim_{n\to\infty} \frac{\frac{6}{n}}{3+\frac{1}{n^2}} = \frac{0}{3+0} = 0$$

7. Therefore, $\ln\, L = 0$, which means $L = e^0 = 1$.
8. **Conclusion:** $\lim_{n\to\infty}(3n^2+1)^{1/n} = 1$.

(b) Evaluate $\lim\, \ln{(a_n)}$ **and interpret the result.**

1. From part (a), we computed $\lim_{n\to\infty} \ln{(a_n)} = 0$.
2. **Interpretation:** Since the natural logarithm function is continuous, we can use the property:

$$\lim_{n\to\infty} \ln{(a_n)} = \ln{(\lim_{n\to\infty} a_n)}$$

3. Our calculation confirms this property: we found $\lim_{n\to\infty} \ln{(a_n)} = 0$ and then concluded that $\lim_{n\to\infty} a_n = e^0 = 1$.

(c) Compute $\lim\, (e^{a_n}-1)$**, with justification.**

1. From part (a), we know $\lim_{n\to\infty} a_n = 1$.
2. The function $f(x) = e^x - 1$ is continuous for all real x.
3. By the Continuous Function Theorem, the limit of a continuous function of a sequence is the function of the limit of the sequence.
4. Therefore:

$$\lim_{n\to\infty} (e^{a_n}-1) = e^{\lim_{n\to\infty} a_n} - 1 = e^1 - 1$$

5. **Conclusion:** $\lim_{n\to\infty}(e^{a_n}-1) = e - 1$.

Q4. Telescoping Series

Answer:

(a) $\sum_{n=1}^{\infty}(\frac{1}{n+3}-\frac{1}{n+4})$

1. Write out the first few partial sums, S_k:

$$S_1 = (\frac{1}{4}-\frac{1}{5})$$
$$S_2 = (\frac{1}{4}-\frac{1}{5}) + (\frac{1}{5}-\frac{1}{6}) = \frac{1}{4}-\frac{1}{6}$$
$$S_3 = (\frac{1}{4}-\frac{1}{5}) + (\frac{1}{5}-\frac{1}{6}) + (\frac{1}{6}-\frac{1}{7}) = \frac{1}{4}-\frac{1}{7}$$
$$S_k = (\frac{1}{4}-\frac{1}{5}) + (\frac{1}{5}-\frac{1}{6}) + \cdots + (\frac{1}{k+3}-\frac{1}{k+4}) = \frac{1}{4}-\frac{1}{k+4}$$

2. The sum of the series is the limit of the partial sums:

$$\sum_{n=1}^{\infty} a_n = \lim_{k\to\infty} S_k = \lim_{k\to\infty}(\frac{1}{4}-\frac{1}{k+4})$$

3. Evaluate the limit:

$$\lim_{k\to\infty}(\frac{1}{4}-\frac{1}{k+4}) = \frac{1}{4}-0 = \frac{1}{4}$$

4. **Conclusion:** The series converges. Its sum is $\frac{1}{4}$.

(b) $\sum_{n=2}^{\infty}(\sqrt{n+1}-\sqrt{n})$

1. Write out the first few partial sums, S_k:

$$S_2 = (\sqrt{3}-\sqrt{2})$$
$$S_3 = (\sqrt{3}-\sqrt{2}) + (\sqrt{4}-\sqrt{3}) = \sqrt{4}-\sqrt{2}$$
$$S_4 = (\sqrt{3}-\sqrt{2}) + (\sqrt{4}-\sqrt{3}) + (\sqrt{5}-\sqrt{4}) = \sqrt{5}-\sqrt{2}$$
$$S_k = (\sqrt{3}-\sqrt{2}) + (\sqrt{4}-\sqrt{3}) + \cdots + (\sqrt{k+1}-\sqrt{k}) = \sqrt{k+1}-\sqrt{2}$$

2. The sum of the series is the limit of the partial sums:

$$\sum_{n=2}^{\infty} a_n = \lim_{k\to\infty} S_k = \lim_{k\to\infty} (\sqrt{k+1} - \sqrt{2})$$

3. Evaluate the limit:

$$\lim_{k\to\infty} (\sqrt{k+1} - \sqrt{2}) = \infty$$

4. **Conclusion:** The series diverges.

Q5. Geometric-Type Series

Answer:

(a) $\sum_{n=1}^{\infty} \frac{2^n}{5^{n+2}}$

1. Simplify the general term:

$$a_n = \frac{2^n}{5^{n+2}} = \frac{2^n}{5^n \cdot 5^2} = \frac{1}{25} \cdot (\frac{2}{5})^n$$

2. The series is now in the form $\sum_{n=1}^{\infty} ar^n$, where $a = \frac{1}{25} \cdot \frac{2}{5} = \frac{2}{125}$ and $r = \frac{2}{5}$. *(Note: The standard form is $\sum_{n=0}^{\infty} ar^n$. Here, it starts at n=1, so the first term is ar)*.

3. Check the condition for convergence: $|\, r \,| = \frac{2}{5} < 1$. Therefore, the series converges.

4. The sum of a convergent geometric series starting from $n = 1$ is given by $\frac{ar}{1-r}$.

$$\text{Sum} = \frac{\frac{2}{125}}{1 - \frac{2}{5}} = \frac{\frac{2}{125}}{\frac{3}{5}} = \frac{2}{125} \cdot \frac{5}{3} = \frac{2}{75}$$

5. **Conclusion:** The series converges. Its sum is $\frac{2}{75}$.

(b) $\sum_{n=0}^{\infty} (-1)^n (\frac{1}{4})^n$

1. Rewrite the general term:

$$a_n = (-1)^n (\frac{1}{4})^n = (-\frac{1}{4})^n$$

2. The series is in the standard geometric form $\sum_{n=0}^{\infty} \quad ar^n$, where $a = 1$ and $r = -\frac{1}{4}$.
3. Check the condition for convergence: $| r |= \frac{1}{4} < 1$. Therefore, the series converges.
4. The sum of a convergent geometric series is $\frac{a}{1-r}$.

$$\text{Sum} = \frac{1}{1-(-\frac{1}{4})} = \frac{1}{1+\frac{1}{4}} = \frac{1}{\frac{5}{4}} = \frac{4}{5}$$

5. **Conclusion:** The series converges. Its sum is $\frac{4}{5}$.

Q6. nth-Term Test for Divergence

Answer:

(a) $a_n = \frac{4n}{n+2}$

1. Find the limit of the sequence:

$$\lim_{n\to\infty} \frac{4n}{n+2} = \lim_{n\to\infty} \frac{4}{1+\frac{2}{n}} = \frac{4}{1} = 4$$

2. Since $\lim_{n\to\infty} a_n = 4 \neq 0$,
3. **Conclusion:** The series $\sum a_n$ diverges by the nth-Term Test.

(b) $a_n = n\cos(\frac{1}{n})$

1. Find the limit of the sequence. Note that $\lim_{n\to\infty} \cos(1/n) = \cos(0) = 1$.
2. Therefore:

$$\lim_{n\to\infty} n\cos(\frac{1}{n}) = \lim_{n\to\infty} n \cdot 1 = \infty$$

(More formally, it is of the form $\infty \cdot 1$*, which diverges to* ∞*).*

3. Since $\lim_{n\to\infty} a_n$ does not exist (is infinite),
4. **Conclusion:** The series $\sum a_n$ diverges by the nth-Term Test.

(c) $a_n = \frac{3^n}{2^{n+1}}$

1. Simplify the general term:

$$a_n = \frac{3^n}{2^{n+1}} = \frac{1}{2} \cdot (\frac{3}{2})^n$$

2. Find the limit of the sequence:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{2} \cdot (\frac{3}{2})^n$$

Since $\frac{3}{2} > 1$, $(\frac{3}{2})^n \to \infty$.

3. Therefore, $\lim_{n\to\infty} a_n = \infty \neq 0$.
4. **Conclusion:** The series $\sum a_n$ diverges by the nth-Term Test.

Q7. Introductory Integral Test

Answer:

(I) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

1. Define the function $f(x) = \frac{1}{x(\ln x)^2}$. This function is positive, continuous, and decreasing for $x \geq 2$.
2. Evaluate the improper integral:

$$\int_2^{\infty} \frac{1}{x(\ln x)^2} dx$$

3. Use substitution: Let $u = \ln x$, so $du = \frac{1}{x} dx$.
   - When $x = 2$, $u = \ln 2$.
   - When $x \to \infty$, $u \to \infty$.
4. The integral becomes:

$$\int_{\ln 2}^{\infty} \frac{1}{u^2} du = \int_{\ln 2}^{\infty} u^{-2} du$$

5. Find the antiderivative:

$$\int u^{-2} du = -u^{-1} = -\frac{1}{u}$$

6. Evaluate the limit:

$$\lim_{b\to\infty} [-\frac{1}{u}]_{\ln 2}^{b} = \lim_{b\to\infty} (-\frac{1}{b} + \frac{1}{\ln 2}) = 0 + \frac{1}{\ln 2} = \frac{1}{\ln 2}$$

7. Since the improper integral converges to a finite value,
8. **Conclusion:** The series $\sum_{n=2}^{\infty} \frac{1}{n(\ln\ n)^2}$ converges.

(II) $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$

1. Define the function $f(x) = \frac{1}{x\sqrt{\ln x}}$. This function is positive, continuous, and decreasing for $x > 1$.
2. Evaluate the improper integral:

$$\int_{2}^{\infty} \frac{1}{x\sqrt{\ln x}} dx$$

3. Use substitution: Let $u = \ln\ x$, so $du = \frac{1}{x} dx$.
   - When $x = 2$, $u = \ln\ 2$.
   - When $x \to \infty$, $u \to \infty$.
4. The integral becomes:

$$\int_{\ln}^{\infty} \frac{1}{\sqrt{u}} du = \int_{\ln}^{\infty} u^{-1/2} du$$

5. Find the antiderivative:

$$\int u^{-1/2} du = 2u^{1/2} = 2\sqrt{u}$$

6. Evaluate the limit:

$$\lim_{b\to\infty}[2\sqrt{u}]_{\ln}^{b} = \lim_{b\to\infty}(2\sqrt{b} - 2\sqrt{\ln 2}) = \infty$$

7. Since the improper integral diverges,
8. **Conclusion:** The series $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ diverges.

****