Subject Name & Code:

MATHEMATICS I- BE01R00041

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TWA – 7

Topic: (Infinite Series: Comparison, Ratio, Root & Integral Tests)

Q-1: p-Series & Integral Test

Answer:

(a) $\sum_{n=1}^{\infty} \frac{1}{n^{1.2}}$

1. This is a p-series of the form $\sum \frac{1}{n^p}$.
2. Here, $p = 1.2$.
3. A p-series converges if $p > 1$ and diverges if $p \leq 1$.
4. Since $1.2 > 1$,
5. **Conclusion:** The series converges.

(b) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$

1. Use the Integral Test. Define $f(x) = \frac{1}{x(\ln x)^3}$. This function is positive, continuous, and decreasing for $x \geq 2$.
2. Evaluate the improper integral:

$$\int_2^{\infty} \frac{1}{x(\ln x)^3} dx$$

3. Use substitution: Let $u = \ln x$, so $du = \frac{1}{x} dx$.
   - When $x = 2$, $u = \ln 2$.
   - When $x \to \infty$, $u \to \infty$.
4. The integral becomes:

$$\int_{\ln}^{\infty} u^{-3} du$$

5. Find the antiderivative:

$$\int u^{-3}du = \frac{u^{-2}}{-2} = -\frac{1}{2u^2}$$

6. Evaluate the limit:

$$\lim_{b\to\infty}[-\frac{1}{2u^2}]_{\ln}^{b} \quad = \lim_{b\to\infty}(-\frac{1}{2b^2} + \frac{1}{2(\ln\ 2)^2}) = 0 + \frac{1}{2(\ln\ 2)^2}$$

7. Since the improper integral converges to a finite value,
8. **Conclusion:** The series converges.

(c) $\sum_{n=2}^{\infty} \quad \frac{1}{\sqrt{n}\ln\ n}$

1. Use the Integral Test. Define $f(x) = \frac{1}{\sqrt{x}\ln\ x}$. This function is positive, continuous, and decreasing for $x \geq 2$.
2. Evaluate the improper integral:

$$\int_2^{\infty} \quad \frac{1}{\sqrt{x}\ln\ x}dx$$

3. Consider the behavior. For large x, $\frac{1}{\sqrt{x}\ln\ x} > \frac{1}{x}$.
   A more formal integral test is needed.
4. Use substitution: Let $u = \ln\ x$, so $x = e^u$ and $dx = e^u du$. Also, $\sqrt{x} = e^{u/2}$.
   - When $x = 2$, $u = \ln\ 2$.
   - When $x \to \infty$, $u \to \infty$.
5. The integral becomes:

$$\int_{\ln 2}^{\infty} \quad \frac{1}{e^{u/2} \cdot u} \cdot e^u du = \int_{\ln}^{\infty} \quad \frac{e^{u/2}}{u}du = \int_{\ln 2}^{\infty} \quad \frac{e^{u/2}}{u}du$$

6. For $u \geq \ln\ 2$, we have $\frac{e^{u/2}}{u} \geq \frac{1}{u}$. Since $\int_{\ln 2}^{\infty} \quad \frac{1}{u}du$ diverges, by the Comparison Theorem for integrals, the larger integral $\int_{\ln}^{\infty} \quad \frac{e^{u/2}}{u}du$ also diverges.
7. **Conclusion:** The series diverges.

Q2. Direct Comparison & Limit Comparison Tests

Answer:

(a) $\sum_{n=1}^{\infty} \frac{5n+3}{n^3+4}$

1. For large n, the dominant terms are $5n$ in the numerator and n^3 in the denominator.
2. $a_n = \frac{5n+3}{n^3+4} \approx \frac{5n}{n^3} = \frac{5}{n^2}$
3. Use the Limit Comparison Test with $b_n = \frac{1}{n^2}$.
4. Compute the limit:

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{5n+3}{n^3+4}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{(5n+3)n^2}{n^3+4} = \lim_{n\to\infty} \frac{5n^3+3n^2}{n^3+4} = \lim_{n\to\infty} \frac{5+\frac{3}{n}}{1+\frac{4}{n^3}}$$
$$= 5$$

5. Since $0 < L < \infty$ and $\sum b_n = \sum \frac{1}{n^2}$ is a convergent p-series ($p = 2 > 1$),
6. **Conclusion:** The series $\sum a_n$ converges.

(b) $\sum_{n=1}^{\infty} \frac{\sqrt{n}+\sin\, n}{n^{3/2}}$

1. Simplify the general term:

$$a_n = \frac{\sqrt{n}+\sin\, n}{n^{3/2}} = \frac{\sqrt{n}}{n^{3/2}} + \frac{\sin n}{n^{3/2}} = \frac{1}{n} + \frac{\sin n}{n^{3/2}}$$

2. Consider the series $\sum \frac{1}{n}$ and $\sum \frac{\sin n}{n^{3/2}}$.
3. The series $\sum \frac{1}{n}$ diverges (harmonic series).
   The series $\sum \frac{\sin n}{n^{3/2}}$ converges by the Comparison Test because $| \frac{\text{si}}{n^{3/2}} | \leq \frac{1}{n^{3/2}}$, and $\sum \frac{1}{n^{3/2}}$ is a convergent p-series ($p = 1.5 > 1$).
4. The sum of a convergent series and a divergent series is divergent.
5. **Conclusion:** The series $\sum a_n$ diverges.

(c) $\sum_{n=1}^{\infty} \frac{2n^3+3}{7n^3-9n+4}$

1. For large n, the dominant terms are $2n^3$ in the numerator and $7n^3$ in the denominator.
2. $a_n = \frac{2n^3+3}{7n^3-9n+4} \approx \frac{2n^3}{7n^3} = \frac{2}{7}$
3. Apply the nth-Term Test for Divergence:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n^3+3}{7n^3-9n+4} = \lim_{n\to\infty} \frac{2+\frac{3}{n^3}}{7-\frac{9}{n^2}+\frac{4}{n^3}} = \frac{2}{7} \neq 0$$

4. Since $\lim_{n\to\infty} a_n \neq 0$,
5. **Conclusion:** The series diverges.

(d) $\sum_{n=2}^{\infty} \frac{\ln(n+1)}{n^2}$

1. For $n \geq 2$, we know $\ln(n+1) < n$. (A stronger, but simpler bound is that $\ln(n+1) < n^p$ for any $p > 0$ for large n. A standard bound is $\ln(n+1) < n^{1/2}$ for large n).
2. Let's use $\ln(n+1) < n^{1/2}$ for $n \geq 1$.
3. Then,

$$a_n = \frac{\ln(n+1)}{n^2} < \frac{n^{1/2}}{n^2} = \frac{1}{n^{3/2}}$$

4. The series $\sum \frac{1}{n^{3/2}}$ is a convergent p-series ($p = 1.5 > 1$).
5. By the Direct Comparison Test, since $0 \leq a_n \leq \frac{1}{n^{3/2}}$ for all $n \geq 2$,
6. **Conclusion:** The series $\sum a_n$ converges.

Q3. Ratio Test

Answer:

(a) $\sum_{n=1}^{\infty} \frac{n!}{4^n}$

1. Let $a_n = \frac{n!}{4^n}$.
2. Form the ratio $|\frac{a_{n+1}}{a_n}|$:

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)!}{4^{n+1}}}{\frac{n!}{4^n}} = \frac{(n+1)!}{n!} \cdot \frac{4^n}{4^{n+1}} = (n+1) \cdot \frac{1}{4}$$

3. Take the limit:

$$L = \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} \frac{n+1}{4} = \infty$$

4. Since $L = \infty > 1$,
5. **Conclusion:** The series diverges.

(b) $\sum_{n=1}^{\infty} \frac{3^n}{n!}$

1. Let $a_n = \frac{3^n}{n!}$.
2. Form the ratio $|\frac{a_{n+1}}{a_n}|$:

$$\frac{a_{n+1}}{a_n} = \frac{\frac{3^{n+1}}{(n+1)!}}{\frac{3^n}{n!}} = \frac{3^{n+1}}{3^n} \cdot \frac{n!}{(n+1)!} = 3 \cdot \frac{1}{n+1}$$

3. Take the limit:

$$L = \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} \frac{3}{n+1} = 0$$

4. Since $L = 0 < 1$,
5. **Conclusion:** The series converges.

(c) $\sum_{n=1}^{\infty} \frac{(2n)!}{n^{2n}}$

1. Let $a_n = \frac{(2n)!}{n^{2n}}$.
2. Form the ratio $|\frac{a_{n+1}}{a_n}|$:
3. Simplify the first term and recognize the limit for the second:

$$\frac{(2n+2)(2n+1)}{(n+1)^2} = \frac{4n^2+6n+2}{n^2+2n+1} \to 4 \text{as } n \to \infty$$
$$(\frac{n}{n+1})^{2n} = ((1-\frac{1}{n+1})^{n+1})^{\frac{2n}{n+1}} \to (e^{-1})^2 = e^{-2} \text{as } n \to \infty$$

4. Therefore, the limit is:

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 4 \cdot e^{-2} = \frac{4}{e^2}$$

5. Since $\frac{4}{e^2} \approx 0.54 < 1$,
6. **Conclusion:** The series converges.

Q4. Root Test

Answer:

(a) $\sum_{n=1}^{\infty} \quad (\frac{n}{n+5})^n$

1. Let $a_n = (\frac{n}{n+5})^n$.
2. Take the nth root:

$$\sqrt[n]{|\, a_n \,|} = \sqrt[n]{(\frac{n}{n+5})^n} = \frac{n}{n+5}$$

3. Take the limit:

$$L = \lim_{n\to\infty} \sqrt[n]{|\, a_n \,|} = \lim_{n\to\infty} \frac{n}{n+5} = \lim_{n\to\infty} \frac{1}{1+\frac{5}{n}} = 1$$

4. Since $L = 1$, the Root Test is inconclusive.
5. Check the nth-Term Test:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (\frac{n}{n+5})^n = \lim_{n\to\infty} (1-\frac{5}{n+5})^n = \lim_{n\to\infty} ((1-\frac{5}{n+5})^{n+5})^{\frac{n}{n+5}}$$
$$= (e^{-5})^1 = e^{-5} \neq 0$$

6. Since $\lim_{n\to\infty} a_n \neq 0$,

7. **Conclusion:** The series diverges.

(b) $\sum_{n=1}^{\infty} \quad (1+\frac{2}{n})^n$

1. Let $a_n = (1+\frac{2}{n})^n$.
2. Take the nth root:

$$\sqrt[n]{|a_n|} = \sqrt[n]{(1+\frac{2}{n})^n} = 1+\frac{2}{n}$$

3. Take the limit:

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} (1+\frac{2}{n}) = 1$$

4. The Root Test is inconclusive.
5. Check the nth-Term Test:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (1+\frac{2}{n})^n = e^2 \neq 0$$

6. Since $\lim_{n\to\infty} a_n \neq 0$,
7. **Conclusion:** The series diverges.

(c) $\sum_{n=1}^{\infty} \quad \frac{n}{(3+s \quad n)^n}$

1. Let $a_n = \frac{n}{(3+\sin\ n)^n}$.
2. Take the nth root:

$$\sqrt[n]{|a_n|} = \sqrt[n]{\frac{n}{(3+\sin\ n)^n}} = \frac{\sqrt[n]{n}}{3+\sin\ n}$$

3. We know $\lim_{n\to\infty} \sqrt[n]{n} = 1$.
   Also, since $-1 \leq \sin\ n \leq 1$, we have $2 \leq 3+\sin\ n \leq 4$.
4. Therefore, we can bound the limit:

$$\frac{\sqrt[n]{n}}{4} \leq \sqrt[n]{|a_n|} \leq \frac{\sqrt[n]{n}}{2}$$

Taking the limit as $n \to \infty$:

$$\frac{1}{4} \le \lim_{n\to\infty} \sqrt[n]{|a_n|} \le \frac{1}{2}$$

5. Since the limit L satisfies $L \le \frac{1}{2} < 1$,
6. **Conclusion:** The series converges.

Q5. Alternating Series & Absolute Convergence

Answer:

(a) $\sum_{n=2}^{\infty} \quad (-1)^{n-1} \frac{1}{n\ln\, n}$

1. **Absolute Convergence:** Consider the absolute series $\sum_{n=2}^{\infty} \quad \frac{1}{n\ln\, n}$.
2. Use the Integral Test with $f(x) = \frac{1}{x\ln\, x}$.
3. $\int_2^{\infty} \quad \frac{1}{xl \quad x} dx$. Let $u = \ln\, x$, $du = \frac{1}{x} dx$.
4. $\int_{\ln 2}^{\infty} \quad \frac{1}{u} du = \lim_{b\to\infty} [\ln\, u]_{\ln}^{b} \quad = \lim_{b\to\infty} (\ln\, b - \ln\,(\ln\, 2)) = \infty$.
5. The integral diverges, so the absolute series diverges. The series is **not absolutely convergent**.
6. **Conditional Convergence:** Check the Alternating Series Test (AST).
   - The series is alternating: $b_n = \frac{1}{n\ln\, n}$.
   - $b_n > 0$ for $n \ge 2$.
   - $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{nl \quad n} = 0$.
   - Check if b_n is decreasing. Consider $f(x) = \frac{1}{xl \quad x}$. $f'(x) = -\frac{\ln\, x+1}{(x\ln\, x)^2} < 0$ for $x > 1$. So b_n is decreasing.
7. All conditions of the AST are satisfied.
8. **Conclusion:** The series converges conditionally.

(b) $\sum_{n=1}^{\infty} \quad (-1)^n \frac{\ln n}{n^{3/2}}$

1. **Absolute Convergence:** Consider the absolute series $\sum_{n=1}^{\infty} \quad \frac{\ln n}{n^{3/2}}$.
2. We use the Comparison Test. As shown in Q2(d), $\frac{\ln n}{n^{3/2}} < \frac{1}{n^{5/4}}$ for large n (since any power n^p grows faster than $\ln n$).
   More formally, $\lim_{n\to\infty} \frac{\ln n/n^{3/2}}{1/n^{5/4}} = \lim_{n\to\infty} \frac{\ln n}{n^{1/4}} = 0$.
   Since $\sum \frac{1}{n^{5/4}}$ converges (p-series, p=1.25>1), the Limit Comparison Test implies $\sum \frac{\ln n}{n^{3/2}}$ converges.
3. Since the absolute series converges,
4. **Conclusion:** The series converges absolutely.

(c) Create one alternating series that converges conditionally but not absolutely.

1. The standard example is the Alternating Harmonic Series.
2. Series: $\sum_{n=1}^{\infty} \quad (-1)^{n-1} \frac{1}{n}$.
3. **Absolute Convergence:** The absolute series is $\sum_{n=1}^{\infty} \quad \frac{1}{n}$, the harmonic series, which diverges.
4. **Conditional Convergence:** It satisfies the Alternating Series Test:
   - $b_n = \frac{1}{n} > 0$
   - $\lim_{n\to\infty} b_n = 0$
   - $b_n = \frac{1}{n}$ is decreasing.
5. Therefore, $\sum_{n=1}^{\infty} \quad (-1)^{n-1} \frac{1}{n}$ converges conditionally.

Q6. Mixed Problems

Answer:

(a) $\sum_{n=1}^{\infty} \quad \frac{2^n}{n^3}$

1. The term has exponential growth in the numerator (2^n) and polynomial growth in the denominator (n^3).

2. Use the Ratio Test. Let $a_n = \frac{2^n}{n^3}$.
3. $\frac{a_{n+1}}{a_n} = \frac{\frac{2^{n+1}}{(n+1)^3}}{\frac{2^n}{n^3}} = 2 \cdot (\frac{n}{n+1})^3$.
4. $L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 2 \cdot 1^3 = 2$.
5. Since $L = 2 > 1$,
6. **Conclusion:** The series diverges.

(b) $\sum_{n=1}^{\infty} \frac{n^4}{5^n}$

1. The term has polynomial growth in the numerator (n^4) and exponential decay in the denominator (5^n).
2. Use the Ratio Test. Let $a_n = \frac{n^4}{5^n}$.
3. $\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)^4}{5^{n+1}}}{\frac{n^4}{5^n}} = \frac{1}{5} \cdot (\frac{n+1}{n})^4$.
4. $L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{5} \cdot 1^4 = \frac{1}{5}$.
5. Since $L = \frac{1}{5} < 1$,
6. **Conclusion:** The series converges.

(c) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)(\ln(\ln n))}$

1. Use the Integral Test. Define $f(x) = \frac{1}{xl \quad x\ln(\ln x)}$. This function is positive, continuous, and decreasing for $x > e$.
2. Evaluate the improper integral:

$$\int_2^{\infty} \frac{1}{x\ln x\ln(\ln x)} dx$$

3. Use substitution: Let $u = \ln(\ln x)$. Then $du = \frac{1}{x\ln x} dx$.
   - When $x = 2$, $u = \ln(\ln 2)$.
   - When $x \to \infty$, $u \to \infty$.
4. The integral becomes:

$$\int_{\ln(\ln 2)}^{\infty} \frac{1}{u} du$$

5. This is a divergent integral (it is the harmonic integral).
6. **Conclusion:** The series diverges.

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