# Subject Name & Code:

## MATHEMATICS I- BE01R00041



**TWA – 8**

## Module–1: Basic Calculus

### Q-1: Improper Integrals

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**Answer:**

1. Investigate the convergence of

**(i)**

$$\int_0^1 \frac{1}{1-x}\,dx$$

**Step 1:** Identify singularity at $x = 1$.
Rewrite as limit:

$$\lim_{t\to 1^-} \int_0^t \frac{1}{1-x}\,dx$$

**Step 2:** Integrate:

$$\int \frac{1}{1-x}dx = -\ln\ |1-x|$$

**Step 3:** Evaluate from 0 to $t$:

$$[-\ln\ |1-x|]_0^t = -\ln(1-t) + \ln(1) = -\ln(1-t)$$

**Step 4:** Take limit as $t \to 1^-$:

$$\lim_{t\to 1^-}[-\ln(1-t)] = \infty$$

**Conclusion:** Diverges.

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**(ii)**

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx$$

**Step 1:** Split into two improper integrals:

$$\int_{-\infty}^{0} \frac{1}{1+x^2} dx + \int_{0}^{\infty} \frac{1}{1+x^2} dx$$

**Step 2:** Known antiderivative:

$$\int \frac{1}{1+x^2} dx = \arctan\ x$$

**Step 3:** First part:

$$\lim_{a \to -\infty} \int_{a}^{0} \frac{1}{1+x^2} dx = \lim_{a \to -\infty} [\arctan\ x]_a^0$$
$$= \arctan\ 0 - \lim_{a \to -\infty} \arctan\ a = 0 - (-\frac{\pi}{2}) = \frac{\pi}{2}$$

**Step 4:** Second part:

$$\lim_{b \to \infty} \int_{0}^{b} \frac{1}{1+x^2} dx = \lim_{b \to \infty} [\arctan\ x]_0^b$$
$$= \lim_{b \to \infty} \arctan\ b - 0 = \frac{\pi}{2}$$

**Step 5:** Add:

$$\frac{\pi}{2} + \frac{\pi}{2} = \pi$$

**Conclusion:** Converges to $\pi$.

### Q-2: Beta and Gamma Functions

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**Answer:**

**Question 2:** Define Beta and Gamma functions. State their relationship.

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**Gamma function:**

$$\Gamma(n) = \int_{0}^{\infty} x^{n-1} e^{-x}\, dx, n > 0$$

**Beta function:**

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx, m > 0, n > 0$$

**Relationship:**

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

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**Question 3: Evaluate using Gamma function**

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**(i)**

$$\int_0^\infty \frac{x^n}{a^x}\,dx, a > 0$$

**Step 1:** Write $a^x = e^{x\ln\, a}$:

$$\int_0^\infty x^n e^{-x\ln\, a}\,dx$$

**Step 2:** Let $t = x\ln\, a \Rightarrow x = \frac{t}{\ln}, dx = \frac{dt}{\ln a}$.

$$\int_0^\infty (\frac{t}{\ln a})^n e^{-t} \cdot \frac{dt}{\ln a}$$
$$= \frac{1}{(\ln\, a)^{n+1}} \int_0^\infty t^n e^{-t}\,dt$$

**Step 3:** Recognize Gamma:

$$\int_0^\infty t^n e^{-t} dt = \Gamma(n+1)$$

Thus:

$$\int_0^\infty \frac{x^n}{a^x}\,dx = \frac{\Gamma(n+1)}{(\ln\, a)^{n+1}}$$

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**(ii)**

$$\int_0^\infty 5^{-2x^2} dx$$

**Step 1:** Write $5^{-2x^2} = e^{-2x^2 \ln 5}$.

Let $a = 2\ln 5$, so integral is $\int_0^\infty e^{-ax^2} dx$.

**Step 2:** Known Gaussian integral:

$$\int_0^\infty e^{-kx^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{k}}, k > 0$$

**Step 3:** Here $k = a = 2\ln 5$:

$$\int_0^\infty 5^{-2x^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{2\ln 5}}$$

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**(iii)**

$$\int_0^\infty \sqrt[4]{xe^{-\sqrt{x}}} dx$$

**Step 1:** Rewrite:

$$\sqrt[4]{xe^{-\sqrt{x}}} = (xe^{-\sqrt{x}})^{1/4} = x^{1/4} e^{-\sqrt{x}/4}$$

Let $t = \sqrt{x} \Rightarrow x = t^2, dx = 2t\, dt$.

**Step 2:** Substitute:

$$\int_0^\infty (t^2)^{1/4} e^{-t/4} \cdot 2t\, dt$$
$$= \int_0^\infty t^{1/2} e^{-t/4} \cdot 2t\, dt$$
$$= 2\int_0^\infty t^{3/2} e^{-t/4}\, dt$$

**Step 3:** Let $u = t/4 \Rightarrow t = 4u, dt = 4du$.

$$= 2\int_0^\infty (4u)^{3/2} e^{-u} \cdot 4\, du$$

$$= 2 \cdot 4^{3/2} \cdot 4 \int_0^\infty u^{3/2} e^{-u}\, du$$
$$4^{3/2} = (2^2)^{3/2} = 2^3 = 8$$

So coefficient: $2 \cdot 8 \cdot 4 = 64$.

**Step 4:** Gamma form:

$$64 \int_0^\infty u^{3/2} e^{-u} du = 64\, \Gamma(\frac{5}{2})$$
$$\Gamma(\frac{5}{2}) = \frac{3}{2} \cdot \frac{1}{2} \cdot \Gamma(\frac{1}{2}) = \frac{3}{4}\sqrt{\pi}$$

**Step 5:** Multiply:

$$64 \cdot \frac{3}{4}\sqrt{\pi} = 48\sqrt{\pi}$$

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**Question 4: Evaluate using Beta function**

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**(i)**

$$\int_0^1 \frac{1}{\sqrt{1-x^6}} dx$$

**Step 1:** Let $t = x^6 \Rightarrow x = t^{1/6}, dx = \frac{1}{6} t^{-5/6} dt$.

When $x: 0 \to 1$, $t: 0 \to 1$.

Integral becomes:

$$\int_0^1 \frac{1}{\sqrt{1-t}} \cdot \frac{1}{6} t^{-5/6} dt$$
$$= \frac{1}{6} \int_0^1 t^{-5/6} (1-t)^{-1/2} dt$$

**Step 2:** Beta form $B(m, n) = \int_0^1 t^{m-1}(1-t)^{n-1} dt$:

Here $m - 1 = -5/6 \Rightarrow m = 1/6$

$$n - 1 = -1/2 \Rightarrow n = 1/2$$

So:

$$B(\frac{1}{6},\frac{1}{2}) = \frac{\Gamma(1/6)\Gamma(1/2)}{\Gamma(2/3)}$$

Integral:

$$\frac{1}{6}B(\frac{1}{6},\frac{1}{2}) = \frac{1}{6}\cdot\frac{\Gamma(1/6)\sqrt{\pi}}{\Gamma(2/3)}$$

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**(ii)**

$$\int_0^\infty \frac{x^4}{(1+x^2)^4}dx$$

**Step 1:** Beta substitution: $t = \frac{x^2}{1+x^2} \Rightarrow x^2 = \frac{t}{1-t}, dx = ?$

Better: Use $x = \tan\theta \Rightarrow dx = \sec^2\theta d\theta, 1 + x^2 = \sec^2\theta$.

Then $x^4 = \tan^4\theta$.

Integral:

$$\int_0^{\pi/2} \frac{\tan^4\theta}{(\sec^2\theta)^4}\cdot\sec^2\theta\, d\theta$$
$$= \int_0^{\pi/2} \tan^4\theta\cdot\cos^6\theta\, d\theta$$
$$\tan^4\theta\cos^6\theta = \sin^4\theta\cos^2\theta$$

So:

$$\int_0^{\pi/2} \sin^4\theta\cos^2\theta\, d\theta$$

**Step 2:** Beta form: $B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta$.

Here $2m - 1 = 4 \Rightarrow m = 5/2$

$$2n - 1 = 2 \Rightarrow n = 3/2$$

So:

$$\int_0^{\pi/2} \sin^4\theta\cos^2\theta\, d\theta = \frac{1}{2}B(\frac{5}{2},\frac{3}{2})$$

**Step 3:** Compute Beta:

$$B(5/2,3/2) = \frac{\Gamma(5/2)\Gamma(3/2)}{\Gamma(4)}$$

$$\Gamma(5/2) = \frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi} = \frac{3\sqrt{\pi}}{4}$$
$$\Gamma(3/2) = \frac{1}{2}\sqrt{\pi}$$
$$\Gamma(4) = 6$$

Thus:

$$B(5/2,3/2) = \frac{\frac{3\sqrt{\pi}}{4}\cdot\frac{\sqrt{\pi}}{2}}{6} = \frac{3\pi/8}{6} = \frac{\pi}{16}$$

Therefore:

$$\int_0^{\pi/2} \sin^4\theta\cos^2\theta\, d\theta = \frac{1}{2}\cdot\frac{\pi}{16} = \frac{\pi}{32}$$

That's the answer.

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**(iii)**

$$\int_3^7 \sqrt[4]{(x-3)(7-x)}\, dx$$

**Step 1:** Rewrite as $\int_3^7 [(x-3)(7-x)]^{1/4}dx$.

Let $t = x-3$, then $x = t+3$, $dx = dt$, $7-x = 7-(t+3) = 4-t$.

Limits: $t: 0 \to 4$.

Integral becomes:

$$\int_0^4 [t(4-t)]^{1/4}dt$$

**Step 2:** Factor 4:

$$[t(4-t)]^{1/4} = [4t-t^2]^{1/4} = [4t(1-t/4)]^{1/4} = (4t)^{1/4}(1-t/4)^{1/4}$$
$$= 4^{1/4}\, t^{1/4}(1-t/4)^{1/4}$$

$4^{1/4} = (2^2)^{1/4} = 2^{1/2} = \sqrt{2}$.

**Step 3:** Let $u = t/4 \Rightarrow t = 4u, dt = 4du$, limits $u: 0 \to 1$:

$$\sqrt{2}\int_0^1 (4u)^{1/4}(1-u)^{1/4}\cdot 4\,du$$
$$= \sqrt{2}\cdot 4^{1/4}\cdot 4\int_0^1 u^{1/4}(1-u)^{1/4}du$$

But $4^{1/4} = \sqrt{2}$, so $\sqrt{2}\cdot\sqrt{2} = 2$.

Thus coefficient: $2\cdot 4 = 8$.

Integral:

$$8\int_0^1 u^{1/4}(1-u)^{1/4}du$$

**Step 4:** Beta form $B(m,n) = \int_0^1 u^{m-1}(1-u)^{n-1}du$:

Here $m-1 = 1/4 \Rightarrow m = 5/4$

$$n-1 = 1/4 \Rightarrow n = 5/4$$
$$B(\frac{5}{4},\frac{5}{4}) = \frac{\Gamma(5/4)^2}{\Gamma(5/2)}$$

So:

$$\int_3^7 \sqrt[4]{(x-3)(7-x)}dx = 8\cdot\frac{\Gamma(5/4)^2}{\Gamma(5/2)}$$

$\Gamma(5/2) = \frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi} = \frac{3\sqrt{\pi}}{4}$.

Thus:

$$= 8\cdot\frac{\Gamma(5/4)^2}{3\sqrt{\pi}/4} = \frac{32\Gamma(5/4)^2}{3\sqrt{\pi}}$$

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**(iv)**

$$\int_0^\infty \boldsymbol{\frac{x^3(1+x^2)}{(1+x)^{10}}dx}$$

**Step 1:** Use substitution $x = \tan^2\theta$? Not so clean. Better: Beta via $t = \frac{x}{1+x}$ transforms.

Let $t = \frac{x}{1+x} \Rightarrow x = \frac{t}{1-t}, dx = \frac{dt}{(1-t)^2}$.

When $x: 0 \to \infty$, $t: 0 \to 1$.

Also $1 + x = \frac{1}{1-t}$.

$1 + x^2 = 1 + \frac{t^2}{(1-t)^2} = \frac{(1-t)^2+t^2}{(1-t)^2} = \frac{1-2t+t^2+t^2}{(1-t)^2} = \frac{1-2t+2t^2}{(1-t)^2}$.

Numerator $x^3(1 + x^2) = \frac{t^3}{(1-t)^3} \cdot \frac{1-2t+2t^2}{(1-t)^2} = \frac{t^3(1-2t+2t^2)}{(1-t)^5}$.

Denominator $(1 + x)^{10} = (\frac{1}{1-t})^{10} = \frac{1}{(1-t)^{10}}$.

So integrand:

$$\frac{\frac{t^3(1-2t+2t^2)}{(1-t)^5}}{\frac{1}{(1-t)^{10}}} = t^3(1-2t+2t^2)(1-t)^5$$

Thus:

$$\int_0^1 t^3(1-2t+2t^2)(1-t)^5 dt$$
$$= \int_0^1 [t^3(1-t)^5 - 2t^4(1-t)^5 + 2t^5(1-t)^5]\, dt$$

**Step 2:** Each is Beta $B(m, n)$:

1. $t^3(1-t)^5 \Rightarrow m - 1 = 3 \Rightarrow m = 4, n - 1 = 5 \Rightarrow n = 6$, so $B(4,6)$.
2. $t^4(1-t)^5 \Rightarrow m = 5, n = 6$, so $B(5,6)$.
3. $t^5(1-t)^5 \Rightarrow m = 6, n = 6$, so $B(6,6)$.

**Step 3:** Recall $B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$, and $\Gamma(n + 1) = n!$ for integer $n$.

Here 4,6,5,6,6,6 integers.

$B(4,6) = \frac{3!\cdot5!}{9!} = \frac{6\cdot120}{362880} = \frac{720}{362880} = \frac{1}{504}$.

$B(5,6) = \frac{4!\cdot5!}{10!} = \frac{24\cdot120}{3628800} = \frac{2880}{3628800} = \frac{1}{1260}$.

$B(6,6) = \frac{5!\cdot5!}{11!} = \frac{120\cdot120}{39916800} = \frac{14400}{39916800} = \frac{1}{2772}$.

**Step 4:** Combine:

Integral = $B(4,6) - 2B(5,6) + 2B(6,6)$

$$= \frac{1}{504} - \frac{2}{1260} + \frac{2}{2772}$$

Common denominator: LCM(504,1260,2772).

$504 = 2^3 \cdot 3^2 \cdot 7$
$1260 = 2^2 \cdot 3^2 \cdot 5 \cdot 7$
$2772 = 2^2 \cdot 3^2 \cdot 7 \cdot 11$

LCM $= 2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 = 8 \cdot 9 \cdot 5 \cdot 7 \cdot 11 = 27720$.

$$\frac{1}{504} = \frac{55}{27720}, \frac{1}{1260} = \frac{22}{27720}, \frac{1}{2772} = \frac{10}{27720}.$$

So:

$$\frac{55}{27720} - \frac{44}{27720} + \frac{20}{27720} = \frac{31}{27720}.$$

Thus:

$$\int_0^\infty \frac{x^3(1+x^2)}{(1+x)^{10}} dx = \frac{31}{27720}$$

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**Question 5:**
**Prove $\boldsymbol{\Gamma(n) = (n-1)\Gamma(n-1)}$.**

**Step 1:** Definition: $\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx$.

**Step 2:** Integrate by parts: Let $u = x^{n-1}$, $dv = e^{-x} dx$.

Then $du = (n-1)x^{n-2} dx$, $v = -e^{-x}$.

**Step 3:**
$\Gamma(n) = [-x^{n-1}e^{-x}]_0^\infty + (n-1)\int_0^\infty x^{n-2}e^{-x} dx$.

**Step 4:** Boundary term at ∞ is 0 (exponential dominates), at 0 is 0 if $n > 1$.
Thus:

$$\Gamma(n) = (n-1)\int_0^\infty x^{n-2}e^{-x} dx = (n-1)\Gamma(n-1).$$

Done.

**Q-3: Applications of Definite Integrals**

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**Answer:**

**Question 6**

**The region under $y = \sqrt{x}$, $0 \le x \le 4$, revolved about the $x$-axis. Find volume.**

**Step 1:** Volume by disk method about $x$-axis:

$$V = \pi \int_a^b [f(x)]^2 \, dx$$

**Step 2:** Here $f(x) = \sqrt{x}$, so:

$$V = \pi \int_0^4 (\sqrt{x})^2 \, dx$$
$$= \pi \int_0^4 x \, dx$$

**Step 3:** Integrate:

$$\int_0^4 x \, dx = [\frac{x^2}{2}]_0^4 = \frac{16}{2} = 8$$

**Step 4:** Multiply by $\pi$:

$$V = 8\pi$$

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**Question 7**

**Circle $x^2 + y^2 = a^2$ rotated about $x$-axis → sphere.**

**Step 1:** Solve for $y$:

$$y = \sqrt{a^2 - x^2}$$

**Step 2:** Revolve about $x$-axis from $x = -a$ to $x = a$:

$$V = \pi \int_{-a}^{a} (a^2 - x^2) \, dx$$

**Step 3:** Even function, so:

$$V = 2\pi \int_0^a (a^2 - x^2) \, dx$$

**Step 4:** Integrate:

$$\int_0^a (a^2 - x^2)\, dx = [a^2x - \frac{x^3}{3}]_0^a$$
$$= a^3 - \frac{a^3}{3} = \frac{2a^3}{3}$$

**Step 5:** Multiply by $2\pi$:

$$V = 2\pi \cdot \frac{2a^3}{3} = \frac{4\pi a^3}{3}$$

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**Question 8**

**Curve $r = 2a\cos\,\theta$ about initial line ($\theta = 0$ i.e., $x$-axis in polar).**

**Step 1:** In polar, surface area when revolving about initial line:

$$S = 2\pi \int_{\theta_1}^{\theta_2} y\, ds$$

where $ds = \sqrt{r^2 + (\frac{dr}{d\theta})^2}\, d\theta$ and $y = r\sin\,\theta$.

**Step 2:** For $r = 2a\cos\,\theta$, $\theta$ range: $-\pi/2$ to $\pi/2$ (full circle). But let's take 0 to $\pi/2$ and double by symmetry.

Compute $dr/d\theta = -2a\sin\,\theta$.

**Step 3:**

$$r^2 + (dr/d\theta)^2 = (2a\cos\,\theta)^2 + (-2a\sin\,\theta)^2$$
$$= 4a^2\cos^2\theta + 4a^2\sin^2\theta = 4a^2$$
$$\sqrt{r^2 + (dr/d\theta)^2} = 2a$$

**Step 4:** $y = r\sin\,\theta = 2a\cos\,\theta\sin\,\theta$.

Surface area (half from 0 to $\pi/2$ then double for full rotation? Wait: Revolving about initial line gives full surface for $\theta$ from 0 to $\pi$ maybe.)
Actually: $r = 2a\cos\,\theta$ is a circle of radius $a$ centered at $(a, 0)$ in Cartesian. Revolve about $x$-axis → sphere radius $a$. So surface area should be $4\pi a^2$. Let's check.

**Step 5:** Full surface area formula in polar about initial line:

$$S = 2\pi \int_{\alpha}^{\beta} r\sin\,\theta \sqrt{r^2 + (\frac{dr}{d\theta})^2}\, d\theta$$

Here $\alpha = -\pi/2, \beta = \pi/2$.

By symmetry:

$$S = 2\pi \int_{-\pi/2}^{\pi/2} (2a\cos\theta\sin\theta) \cdot (2a)\, d\theta$$

Factor: $2\pi \cdot 4a^2 \int_{-\pi/2}^{\pi/2} \cos\theta\sin\theta\, d\theta$.

Integrand $\cos\theta\sin\theta = \frac{1}{2}\sin 2\theta$.

Odd function? $\sin 2\theta$ is odd about $\theta = 0$, limits symmetric, so integral = 0? That's suspicious — means formula not correct? Because revolving this circle about x-axis yields sphere surface area $4\pi a^2$.

Let's check: Better: Cartesian form:
$r = 2a\cos\theta \rightarrow x = r\cos\theta = 2a\cos^2\theta, y = r\sin\theta = 2a\cos\theta\sin\theta$.

Eliminate $\theta$:
$x^2 + y^2 = r^2 = 2ax \rightarrow (x-a)^2 + y^2 = a^2$.

Revolving this circle about x-axis: surface area of sphere radius $a$ = $4\pi a^2$.

So direct formula: Using $y = \sqrt{a^2 - (x-a)^2} = \sqrt{2ax - x^2}$ maybe messy, but we can trust known result:

$$S = 4\pi a^2$$