Subject Name & Code:

MATHEMATICS I- BE01R00041

Assignment – 1

Topic: Improper integrals

Question 1: Define Improper integral of First kind and evaluate $\int_0^\infty \frac{dx}{x^2+1}$

Solution:

Step 1: Write the improper integral as a limit.

$$\int_0^\infty \frac{dx}{x^2+1} = \lim_{b\to\infty} \int_0^b \frac{dx}{x^2+1}$$

Step 2: Find the antiderivative.
We know that $\int \frac{dx}{x^2+1} = \tan^{-1}(x) + C$.

Step 3: Evaluate the definite integral from 0 to b.

$$\int_0^b \frac{dx}{x^2+1} = [\tan^{-1}(x)]_0^b = \tan^{-1}(b) - \tan^{-1}(0) = \tan^{-1}(b) - 0$$
$$= \tan^{-1}(b)$$

Step 4: Take the limit as $b \to \infty$.

$$\lim_{b\to\infty} \tan^{-1}(b) = \frac{\pi}{2}$$

Final Answer:

$$\boxed{\frac{\pi}{2}}$$

Question 2, Part (1): Evaluate $\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx$

Solution:

Step 1: Split the integral at zero (or any point) since it's an even function.

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}dx = \int_{-\infty}^{0} \frac{1}{1+x^2}dx + \int_{0}^{\infty} \frac{1}{1+x^2}dx$$

Step 2: Use the limit definition for the first integral.

$$\int_{-\infty}^{0} \frac{1}{1+x^2}dx = \lim_{a\to-\infty}\int_{a}^{0} \frac{1}{1+x^2}dx$$

Step 3: Find the antiderivative.

$$\int \frac{1}{1+x^2}dx = \tan^{-1}(x)$$

Step 4: Evaluate the first definite integral.

$$\lim_{a\to-\infty}[\tan^{-1}(x)]_a^0 = \lim_{a\to-\infty}(\tan^{-1}(0) - \tan^{-1}(a)) = 0 - (-\frac{\pi}{2}) = \frac{\pi}{2}$$

Step 5: We already calculated the second integral in Question 1.

$$\int_{0}^{\infty} \frac{1}{1+x^2}dx = \frac{\pi}{2}$$

Step 6: Add the results.

$$\frac{\pi}{2} + \frac{\pi}{2} = \pi$$

Final Answer for Part (1):

$$\boxed{\pi}$$

Question 2, Part (2): Evaluate $\int_{-\infty}^{\infty} \frac{1}{e^x+e^{-x}}dx$

Solution:

Step 1: Simplify the integrand.

$$\frac{1}{e^x+e^{-x}} = \frac{1}{\frac{e^{2x}+1}{e^x}} = \frac{e^x}{1+e^{2x}}$$

Step 2: Write the integral as a limit.

$$\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}}dx = \lim_{a\to-\infty}\lim_{b\to\infty}\int_a^b \frac{e^x}{1+(e^x)^2}dx$$

Step 3: Use substitution. Let $u = e^x$, then $du = e^x dx$.
When $x = a$, $u = e^a$. When $x = b$, $u = e^b$.

Step 4: Change the limits and integrate.

$$\int \frac{e^x}{1+(e^x)^2}dx = \int \frac{1}{1+u^2}du = \tan^{-1}(u) + C = \tan^{-1}(e^x) + C$$

Step 5: Evaluate the definite integral.

$$\int_a^b \frac{e^x}{1+e^{2x}}dx = [\tan^{-1}(e^x)]_a^b = \tan^{-1}(e^b) - \tan^{-1}(e^a)$$

Step 6: Take the limits.

$$\lim_{b\to\infty}\tan^{-1}(e^b) = \tan^{-1}(\infty) = \frac{\pi}{2}$$
$$\lim_{a\to-\infty}\tan^{-1}(e^a) = \tan^{-1}(0) = 0$$

Step 7: Combine the results.

$$\frac{\pi}{2} - 0 = \frac{\pi}{2}$$

Final Answer for Part (2):

$$\boxed{\frac{\pi}{2}}$$

Question 2, Part (3): Evaluate $\int_0^\infty \frac{dx}{(1+x^2)(1+\tan^{-1} x)}$

Solution:

Step 1: Use substitution. Let $u = \tan^{-1}(x)$, then $du = \frac{1}{1+x^2} dx$.
When $x = 0$, $u = \tan^{-1}(0) = 0$.
When $x \to \infty$, $u \to \frac{\pi}{2}$.

Step 2: Substitute into the integral.

$$\int_0^\infty \frac{1}{(1+x^2)(1+\tan^{-1} x)} dx = \int_0^{\pi/2} \frac{1}{1+u} du$$

Step 3: Integrate with respect to u.

$$\int \frac{1}{1+u} du = \ln |1+u| + C$$

Step 4: Evaluate the definite integral.

$$[\ln |1+u|]_0^{\pi/2} = \ln (1+\frac{\pi}{2}) - \ln (1+0) = \ln (1+\frac{\pi}{2}) - 0$$

Final Answer for Part (3):

$$\boxed{\ln(1+\frac{\pi}{2})}$$

Question 3: Define Improper integral of Second kind and evaluate $\int_0^3 \frac{1}{\sqrt{3-x}} dx$

Solution:

Step 1: Identify the point of discontinuity. The integrand $\frac{1}{\sqrt{3-x}}$ is undefined at $x = 3$.

Step 2: Write the integral as a limit.

$$\int_0^3 \frac{1}{\sqrt{3-x}} dx = \lim_{b \to 3^-} \int_0^b \frac{1}{\sqrt{3-x}} dx$$

Step 3: Use substitution. Let $u = 3 - x$, then $du = -dx$.
When $x = 0, u = 3$. When $x = b, u = 3 - b$.

Step 4: Rewrite and integrate.

$$\int_0^b (3-x)^{-1/2} dx = -\int_3^{3-b} u^{-1/2} du = \int_{3-b}^3 u^{-1/2} du$$
$$\int u^{-1/2} du = 2u^{1/2} + C = 2\sqrt{u} + C$$

Step 5: Evaluate the definite integral.

$$[2\sqrt{u}]_{3-b}^3 = 2\sqrt{3} - 2\sqrt{3-b}$$

Step 6: Take the limit as $b \to 3^-$.

$$\lim_{b \to 3^-} (2\sqrt{3} - 2\sqrt{3-b}) = 2\sqrt{3} - 2\sqrt{0} = 2\sqrt{3}$$

Final Answer:

$$\boxed{2\sqrt{3}}$$

Question 4: Check the convergence of $\int_0^3 \frac{dx}{\sqrt{9-x^2}}$

Solution:

Step 1: Identify the point of discontinuity. The integrand $\frac{1}{\sqrt{9-x^2}}$ is undefined at $x = 3$.

Step 2: Write the integral as a limit.

$$\int_0^3 \frac{dx}{\sqrt{9-x^2}} = \lim_{b \to 3^-} \int_0^b \frac{dx}{\sqrt{9-x^2}}$$

Step 3: Find the antiderivative.

$$\int \frac{dx}{\sqrt{9-x^2}} = \sin^{-1}(\frac{x}{3}) + C$$

Step 4: Evaluate the definite integral from 0 to b.

$$\int_0^b \frac{dx}{\sqrt{9-x^2}} = [\sin^{-1}(\frac{x}{3})]_0^b = \sin^{-1}(\frac{b}{3}) - \sin^{-1}(0) = \sin^{-1}(\frac{b}{3})$$

Step 5: Take the limit as $b \to 3^-$.

$$\lim_{b\to 3^-} \sin^{-1}(\frac{b}{3}) = \sin^{-1}(1) = \frac{\pi}{2}$$

Step 6: Since the limit exists and is finite, the integral converges.

Final Answer:

$$\boxed{\text{Converges}}$$