Assignment – 2

Topic: Beta & Gamma function,

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Question 1, Part (i): Define Gamma function. Show that $\Gamma(n+1) = n\Gamma(n)$

Solution:

Step 1: Write the definition of the Gamma function.

$$\Gamma(n) = \int_0^\infty e^{-t} t^{n-1} dt$$

Step 2: Write the expression for $\Gamma(n+1)$.

$$\Gamma(n+1) = \int_0^\infty e^{-t} t^n dt$$

Step 3: Apply integration by parts. Let $u = t^n$, $dv = e^{-t} dt$.
Then $du = nt^{n-1} dt$, $v = -e^{-t}$.

Step 4: Perform the integration.

$$\Gamma(n+1) = [-t^n e^{-t}]_0^\infty + \int_0^\infty e^{-t} n t^{n-1} dt$$

Step 5: Evaluate the boundary term.
As $t \to \infty$, $-t^n e^{-t} \to 0$.
As $t \to 0$, $-t^n e^{-t} \to 0$.
So, $[-t^n e^{-t}]_0^\infty = 0$.

Step 6: Simplify the remaining integral.

$$\Gamma(n+1) = 0 + n \int_0^\infty e^{-t} t^{n-1} dt = n\Gamma(n)$$

Final Proof:

$$\boxed{\Gamma(n+1) = n\Gamma(n)}$$

Question 1, Part (ii): Show that $\Gamma(n) = 2\int_0^\infty e^{-x^2} x^{2n-1} dx$

Solution:

Step 1: Start with the standard Gamma definition: $\Gamma(n) = \int_0^\infty e^{-t} t^{n-1} dt$.

Step 2: Make the substitution $t = x^2$. Then $dt = 2xdx$, and $t^{n-1} = (x^2)^{n-1} = x^{2n-2}$.

Step 3: Substitute into the Gamma integral.

$$\Gamma(n) = \int_0^\infty e^{-x^2} \cdot x^{2n-2} \cdot 2xdx = 2\int_0^\infty e^{-x^2} x^{2n-1} dx$$

Final Proof:

$$\boxed{\Gamma(n) = 2\int_0^\infty e^{-x^2} x^{2n-1} dx}$$

Question 2, Part (i): Evaluate $\Gamma(\frac{7}{2})$

Solution:

Step 1: Use the recurrence relation of the Gamma function: $\Gamma(n+1) = n\Gamma(n)$.

Step 2: Apply it repeatedly starting from $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

$$\Gamma(\frac{3}{2}) = \frac{1}{2} \cdot \Gamma(\frac{1}{2}) = \frac{1}{2}\sqrt{\pi}$$
$$\Gamma(\frac{5}{2}) = \frac{3}{2} \cdot \Gamma(\frac{3}{2}) = \frac{3}{2} \cdot \frac{1}{2}\sqrt{\pi} = \frac{3}{4}\sqrt{\pi}$$
$$\Gamma(\frac{7}{2}) = \frac{5}{2} \cdot \Gamma(\frac{5}{2}) = \frac{5}{2} \cdot \frac{3}{4}\sqrt{\pi} = \frac{15}{8}\sqrt{\pi}$$

Final Answer:

$$\boxed{\frac{15}{8}\sqrt{\pi}}$$

Question 2, Part (ii): Evaluate $\int_0^\infty \frac{x^5}{5^x} dx$

Solution:

Step 1: Rewrite the integrand. Note that $5^x = e^{\ln(5)x}$.

$$\int_0^\infty \frac{x^5}{5^x} dx = \int_0^\infty x^5 e^{-\ln(5)x} dx$$

Step 2: Recognize the form of the Gamma function: $\Gamma(n) = \int_0^\infty e^{-t} t^{n-1} dt$. Let $t = \ln(5)x$, so $dt = \ln(5)dx$, and $x = \frac{t}{\ln(5)}$.

Step 3: Substitute into the integral.

$$\int_0^\infty x^5 e^{-l\ \ (5)x} dx = \int_0^\infty \left(\frac{t}{\ln(5)}\right)^5 e^{-t} \cdot \frac{dt}{\ln(5)} = \frac{1}{(\ln(5))^6} \int_0^\infty t^5 e^{-t} dt$$

Step 4: Identify the Gamma function. Here, $n - 1 = 5$ so $n = 6$.

$$\int_0^\infty t^5 e^{-t} dt = \Gamma(6) = 5!$$

Step 5: Calculate the final result.

$$\frac{1}{(\ln(5))^6} \cdot 5! = \frac{120}{(\ln 5)^6}$$

Final Answer:

$$\boxed{\frac{120}{(\ln 5)^6}}$$

Question 2, Part (iii): Evaluate $\int_0^1 x^4 (\log x)^4 dx$

Solution:

Step 1: Use the standard form: $\int_0^1 x^{m-1} (\log x)^{n-1} dx = \frac{(-1)^{n-1}\Gamma(n)}{m^n}$.

Step 2: Identify $m - 1 = 4$ so $m = 5$, and $n - 1 = 4$ so $n = 5$.

Step 3: Apply the formula.

$$\int_0^1 x^4(\log x)^4 dx = \frac{(-1)^{5-1}\Gamma(5)}{5^5} = \frac{(+1)\cdot 4!}{5^5}$$

Step 4: Calculate the numerical value.

$$4! = 24, 5^5 = 3125$$
$$\frac{24}{3125}$$

Final Answer:

$$\boxed{\frac{24}{3125}}$$

Question 3: Define Beta function. Prove that $\beta(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta d\theta$

Solution:

Step 1: Write the standard definition of the Beta function.

$$\beta(m,n) = \int_0^1 t^{m-1}(1-t)^{n-1}dt$$

Step 2: Make the trigonometric substitution $t = \sin^2\theta$. Then $dt = 2\sin\theta\cos\theta d\theta$.
Also, $t^{m-1} = (\sin^2\theta)^{m-1} = \sin^{2m-2}\theta$.
And $(1-t)^{n-1} = (1-\sin^2\theta)^{n-1} = (\cos^2\theta)^{n-1} = \cos^{2n-2}\theta$.

Step 3: Substitute into the Beta integral.

$$\beta(m,n) = \int_0^1 t^{m-1}(1-t)^{n-1}dt$$
$$= \int_{\theta=0}^{\pi/2} \sin^{2m-2}\theta \cdot \cos^{2n-2}\theta \cdot 2\sin\theta\cos\theta d\theta$$

Step 4: Simplify the exponents.

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-2+1}\theta\cos^{2n-2+1}\theta d\theta$$
$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta d\theta$$

Final Proof:

$$\boxed{\beta(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\cos^{2n-1}\theta d\theta}$$

Question 4, Part (i): Evaluate $\int_{-1}^{1} (1+x)^4(1-x)^3dx$

Solution:

Step 1: Recognize the Beta function form. Let $t = \frac{1+x}{2}$. Then $1 + x = 2t$, $1 - x = 2(1-t)$, and $dx = 2dt$.
When $x = -1$, $t = 0$. When $x = 1$, $t = 1$.

Step 2: Substitute into the integral.

$$\int_{-1}^{1} (1+x)^4(1-x)^3dx = \int_0^1 (2t)^4 \cdot [2(1-t)]^3 \cdot 2dt$$

Step 3: Simplify the constants.

$$= \int_0^1 16t^4 \cdot 8(1-t)^3 \cdot 2dt = 16 \cdot 8 \cdot 2\int_0^1 t^4(1-t)^3dt$$
$$= 256\int_0^1 t^4(1-t)^3dt$$

Step 4: Identify the Beta function parameters: $m - 1 = 4 \Rightarrow m = 5$, $n - 1 = 3 \Rightarrow n = 4$.

$$\int_0^1 t^4(1-t)^3dt = \beta(5,4) = \frac{\Gamma(5)\Gamma(4)}{\Gamma(5+4)} = \frac{4!\cdot 3!}{8!}$$

Step 5: Calculate the factorials.

$$4! = 24, 3! = 6, 8! = 40320$$

$$\frac{24 \cdot 6}{40320} = \frac{144}{40320} = \frac{1}{280}$$

Step 6: Multiply by the constant factor.

$$256 \cdot \frac{1}{280} = \frac{256}{280} = \frac{32}{35}$$

Final Answer:

$$\boxed{\frac{32}{35}}$$

Question 4, Part (ii): Evaluate $\int_0^{\frac{\pi}{2}} \sin^6\theta \cos^7\theta d\theta$

Solution:

Step 1: Write the integral in the standard Beta form from Question 3.

$$\int_0^{\pi/2} \sin^6\theta \cos^7\theta d\theta = \frac{1}{2}\beta(\frac{2m-1=6 \Rightarrow m=7/2}{2}, \frac{2n-1=7 \Rightarrow n=4}{2})$$

Wait, careful: The standard form is $2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta d\theta = \beta(m,n)$.

So, $2m-1=6 \Rightarrow 2m=7 \Rightarrow m=\frac{7}{2}$.
$2n-1=7 \Rightarrow 2n=8 \Rightarrow n=4$.

Step 2: Apply the formula.

$$\int_0^{\pi/2} \sin^6\theta \cos^7\theta d\theta = \frac{1}{2}\beta(\frac{7}{2}, 4)$$

Step 3: Express the Beta function in terms of Gamma.

$$\beta(\frac{7}{2}, 4) = \frac{\Gamma(7/2)\Gamma(4)}{\Gamma(7/2+4)} = \frac{\Gamma(7/2) \cdot 3!}{\Gamma(15/2)}$$

Step 4: Use the property $\Gamma(z+1) = z\Gamma(z)$ to simplify.

$$\Gamma(15/2) = \frac{13}{2} \cdot \frac{11}{2} \cdot \frac{9}{2} \cdot \frac{7}{2} \cdot \Gamma(7/2)$$

So,

$$\frac{\Gamma(7/2)}{\Gamma(15/2)} = \frac{1}{\frac{13}{2} \cdot \frac{11}{2} \cdot \frac{9}{2} \cdot \frac{7}{2}} = \frac{16}{13 \cdot 11 \cdot 9 \cdot 7}$$

Step 5: Put it all together.

$$\frac{1}{2}\beta(\frac{7}{2}, 4) = \frac{1}{2} \cdot (\frac{\Gamma(7/2) \cdot 6}{\Gamma(15/2)}) = \frac{1}{2} \cdot 6 \cdot \frac{16}{13 \cdot 11 \cdot 9 \cdot 7} = \frac{48}{13 \cdot 11 \cdot 9 \cdot 7}$$

Step 6: Calculate the denominator.
$13 \cdot 11 = 143, 143 \cdot 9 = 1287, 1287 \cdot 7 = 9009.$

Final Answer:

$$\boxed{\frac{48}{9009}}$$