Assignment – 3

Topic: Application of definite integral

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Question 1: The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross-sections perpendicular to the x-axis between these planes are squares whose diagonals run from the semicircle $y = -\sqrt{1-x^2}$ to the semicircle . Find the volume.

Solution:

Step 1: Find the side length of the square.
The diagonal d runs from $y = -\sqrt{1-x^2}$ to , so:

$$d = \sqrt{1-x^2} - (-\sqrt{1-x^2}) = 2\sqrt{1-x^2}$$

For a square, diagonal $d = s\sqrt{2}$, so side $s = \frac{d}{\sqrt{2}} = \frac{2\sqrt{1-x^2}}{\sqrt{2}} = \sqrt{2} \cdot \sqrt{1-x^2}$

Step 2: Write the area of the square cross-section.

$$A(x) = s^2 = (\sqrt{2} \cdot \sqrt{1-x^2})^2 = 2(1-x^2)$$

Step 3: Set up and evaluate the volume integral.

$$V = \int_{-1}^{1} A(x)dx = \int_{-1}^{1} 2(1-x^2)dx$$
$$= 2[x - \frac{x^3}{3}]_{-1}^{1}$$
$$= 2[(1 - \frac{1}{3}) - (-1 + \frac{1}{3})]$$
$$= 2[\frac{2}{3} + \frac{2}{3}] = 2 \cdot \frac{4}{3} = \frac{8}{3}$$

Final Answer:

$$\boxed{\frac{8}{3}}$$

Question 2: Find the volume of the solid generated by revolving the region bounded by $y = x^2$, $y = 0$, $x = 2$ about the x-axis by the disk method.

Solution:

Step 1: Identify the radius.
Radius $r(x) = y = x^2$

Step 2: Write the volume formula for disk method.

$$V = \pi \int_a^b \quad [r(x)]^2 dx = \pi \int_0^2 \quad (x^2)^2 dx = \pi \int_0^2 \quad x^4 dx$$

Step 3: Evaluate the integral.

$$\pi[\frac{x^5}{5}]_0^2 = \pi \cdot \frac{32}{5}$$

Final Answer:

$$\boxed{\frac{32\pi}{5}}$$

Question 3: Find the volume of the solid generated by revolving the regions bounded by $y = x^2 + 1$, $y = x + 3$ about the x-axis by the washer method.

Solution:

Step 1: Find points of intersection.

$$x^2 + 1 = x + 3$$
$$x^2 - x - 2 = 0$$
$$(x - 2)(x + 1) = 0$$
$$x = -1, x = 2$$

Step 2: Identify outer radius $R(x)$ and inner radius $r(x)$.
Outer: $R(x) = x + 3$, Inner: $r(x) = x^2 + 1$

Step 3: Washer volume formula.

$$V = \pi \int_{-1}^{2} [(x + 3)^2 - (x^2 + 1)^2] \, dx$$

Step 4: Expand and simplify the integrand.

$$(x+3)^2 = x^2 + 6x + 9$$
$$(x^2+1)^2 = x^4 + 2x^2 + 1$$
$$(x+3)^2 - (x^2+1)^2 = -x^4 - x^2 + 6x + 8$$

Step 5: Integrate.

$$V = \pi \int_{-1}^{2} (-x^4 - x^2 + 6x + 8)dx$$
$$= \pi[-\frac{x^5}{5} - \frac{x^3}{3} + 3x^2 + 8x]_{-1}^{2}$$

Step 6: Evaluate at bounds.
At $x = 2$:

$$-\frac{32}{5} - \frac{8}{3} + 12 + 16 = -\frac{32}{5} - \frac{8}{3} + 28$$

At $x = -1$:

$$\frac{1}{5} + \frac{1}{3} + 3 - 8 = \frac{1}{5} + \frac{1}{3} - 5$$

Subtract:

$$(-\frac{32}{5} - \frac{8}{3} + 28) - (\frac{1}{5} + \frac{1}{3} - 5)$$
$$= -\frac{33}{5} - 3 + 33$$
$$= -\frac{33}{5} + 30 = \frac{-33+150}{5} = \frac{117}{5}$$
$$V = \frac{117\pi}{5}$$

Final Answer:

$$\boxed{\frac{117\pi}{5}}$$

Question 4: Find the length of the curve $y = \frac{1}{2}(e^x + e^{-x});\ 0 \leq x \leq 2$

Solution:

Step 1: Compute the derivative.

$$y' = \frac{1}{2}(e^x - e^{-x})$$

Step 2: Write the arc length formula.

$$L = \int_0^2 \sqrt{1 + (y')^2}dx$$

$$(y')^2 = \frac{1}{4}(e^{2x} - 2 + e^{-2x})$$

$$1 + (y')^2 = 1 + \frac{1}{4}e^{2x} - \frac{1}{2} + \frac{1}{4}e^{-2x} = \frac{1}{4}e^{2x} + \frac{1}{2} + \frac{1}{4}e^{-2x}$$

$$= \frac{1}{4}(e^{2x} + 2 + e^{-2x}) = \frac{1}{4}(e^x + e^{-x})^2$$

Step 3: Simplify the integrand.

$$\sqrt{1 + (y')^2} = \sqrt{\frac{1}{4}(e^x + e^{-x})^2} = \frac{1}{2}(e^x + e^{-x}) = y$$

Step 4: Integrate.

$$L = \int_0^2 \frac{1}{2}(e^x + e^{-x})dx = \frac{1}{2}[e^x - e^{-x}]_0^2$$

$$= \frac{1}{2}[(e^2 - e^{-2}) - (1 - 1)] = \frac{1}{2}(e^2 - e^{-2})$$

Final Answer:

$$\boxed{\frac{e^2 - e^{-2}}{2}}$$

Question 5: Find the length of asteroid $x^{2/3} + y^{2/3} = a^{2/3}$.

Solution:

Step 1: Use parametric form.
Let $x = a\cos^3 t, y = a\sin^3 t, 0 \leq t \leq 2\pi$.

Step 2: Compute derivatives.

$$\frac{dx}{dt} = -3a\cos^2 t\sin t$$
$$\frac{dy}{dt} = 3a\sin^2 t\cos t$$

Step 3: Arc length formula.

$$L = 4\int_0^{\pi/2} \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 dt}$$
$$= 4\int_0^{\pi/2} \sqrt{9a^2\cos^4 t\sin^2 t + 9a^2\sin^4 t\cos^2 t dt}$$
$$= 4\int_0^{\pi/2} \sqrt{9a^2\cos^2 t\sin^2 t(\cos^2 t + \sin^2 t)}dt$$
$$= 4\int_0^{\pi/2} 3a \mid \cos t\sin t \mid dt$$

Since $0 \leq t \leq \pi/2$, $\cos t\sin t \geq 0$:

$$= 12a\int_0^{\pi/2} \cos t\sin t dt$$

Step 4: Integrate.
Let $u = \sin t, du = \cos t dt$:

$$12a\int_0^1 udu = 12a[\frac{u^2}{2}]_0^1 = 12a \cdot \frac{1}{2} = 6a$$

Final Answer:

$$\boxed{6a}$$

Question 6: Find the area of the surface of revolution of the solid generated by revolving the ellipse $\frac{x^2}{16} + \frac{y^2}{4} = 1$ about the x-axis.

Solution:

Step 1: Solve for y.

$$y = 2\sqrt{1 - \frac{x^2}{16}} = 2 \cdot \frac{\sqrt{16 - x^2}}{4} = \frac{\sqrt{16 - x^2}}{2}$$

Step 2: Compute derivative.

$$\frac{dy}{dx} = \frac{1}{2} \cdot \frac{-x}{\sqrt{16 - x^2}} = -\frac{x}{2\sqrt{16 - x^2}}$$

Step 3: Surface area formula.

$$S = 2\pi \int_{-4}^{4} y\sqrt{1 + (\frac{dy}{dx})^2} dx$$

$$1 + (\frac{dy}{dx})^2 = 1 + \frac{x^2}{4(16 - x^2)} = \frac{4(16 - x^2) + x^2}{4(16 - x^2)} = \frac{64 - 3x^2}{4(16 - x^2)}$$

$$\sqrt{1 + (\frac{dy}{dx})^2} = \frac{\sqrt{64 - 3x^2}}{2\sqrt{16 - x^2}}$$

Step 4: Substitute into integral.

$$S = 2\pi \int_{-4}^{4} \frac{\sqrt{16 - x^2}}{2} \cdot \frac{\sqrt{64 - 3x^2}}{2\sqrt{16 - x^2}} dx$$

$$= \frac{\pi}{2} \int_{-4}^{4} \sqrt{64 - 3x^2} dx$$

Step 5: Even function, so:

$$S = \pi \int_{0}^{4} \sqrt{64 - 3x^2} dx$$

Step 6: Use substitution $x = \frac{8}{\sqrt{3}}\sin\theta$, $dx = \frac{8}{\sqrt{3}}\cos\theta d\theta$.

When $x = 0$, $\theta = 0$; when $x = 4$, $\sin\theta = \frac{4\sqrt{3}}{8} = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{\pi}{3}$.

$$\sqrt{64 - 3x^2} = \sqrt{64 - 3 \cdot \frac{64}{3}\sin^2\theta} = 8\sqrt{1 - \sin^2\theta} = 8\cos\theta$$

$$S = \pi\int_0^{\pi/3} 8\cos\theta \cdot \frac{8}{\sqrt{3}}\cos\theta d\theta = \frac{64\pi}{\sqrt{3}}\int_0^{\pi/3} \cos^2\theta d\theta$$

Step 7: Integrate.

$$\int_0^{\pi/3} \cos^2\theta d\theta = \int_0^{\pi/3} \frac{1 + \cos 2\theta}{2} d\theta = [\frac{\theta}{2} + \frac{\sin 2\theta}{4}]_0^{\pi/3}$$

$$= \frac{\pi}{6} + \frac{\sin(2\pi/3)}{4} = \frac{\pi}{6} + \frac{\sqrt{3}/2}{4} = \frac{\pi}{6} + \frac{\sqrt{3}}{8}$$

$$S = \frac{64\pi}{\sqrt{3}}(\frac{\pi}{6} + \frac{\sqrt{3}}{8}) = \frac{32\pi^2}{3\sqrt{3}} + 8\pi$$

Final Answer:

$$\boxed{\frac{32\pi^2}{3\sqrt{3}} + 8\pi}$$

Question 7: Find the surface area generated by revolving the loop of the curve $9ay^2 = x(3a - x)^2$ about the x-axis.

Solution:

Step 1: Find limits of loop.
Set $y = 0$: $x(3a - x)^2 = 0 \Rightarrow x = 0, 3a$. Loop is from $x = 0$ to $x = 3a$.

Step 2: Solve for y.

$$y = \frac{\sqrt{x}(3a - x)}{3\sqrt{a}}$$

Step 3: Compute derivative.

$$\frac{dy}{dx} = \frac{1}{3\sqrt{a}}[\frac{1}{2\sqrt{x}}(3a - x) - \sqrt{x}]$$

$$= \frac{1}{3\sqrt{a}}[\frac{3a-x}{2\sqrt{x}} - \sqrt{x}] = \frac{1}{3\sqrt{a}} \cdot \frac{3a-x-2x}{2\sqrt{x}} = \frac{1}{3\sqrt{a}} \cdot \frac{3a-3x}{2\sqrt{x}} = \frac{a-x}{2\sqrt{ax}}$$

Step 4: Surface area formula.

$$S = 2\pi \int_0^{3a} y\sqrt{1 + (\frac{dy}{dx})^2} dx$$

$$1 + (\frac{dy}{dx})^2 = 1 + \frac{(a-x)^2}{4ax} = \frac{4ax + a^2 - 2ax + x^2}{4ax} = \frac{a^2 + 2ax + x^2}{4ax}$$

$$= \frac{(a+x)^2}{4ax}$$

$$\sqrt{1 + (\frac{dy}{dx})^2} = \frac{a+x}{2\sqrt{ax}}$$

Step 5: Substitute into integral.

$$S = 2\pi \int_0^{3a} \frac{\sqrt{x}(3a-x)}{3\sqrt{a}} \cdot \frac{a+x}{2\sqrt{ax}} dx$$

$$= \frac{\pi}{3a} \int_0^{3a} (3a-x)(a+x) dx$$

Step 6: Expand and integrate.

$$(3a-x)(a+x) = 3a^2 + 3ax - ax - x^2 = 3a^2 + 2ax - x^2$$

$$\int_0^{3a} (3a^2 + 2ax - x^2) dx = [3a^2x + ax^2 - \frac{x^3}{3}]_0^{3a}$$

$$= 9a^3 + 9a^3 - 9a^3 = 9a^3$$

$$S = \frac{\pi}{3a} \cdot 9a^3 = 3\pi a^2$$

Final Answer:

$$\boxed{3\pi a^2}$$

Question 8: Find the surface area of the solid generated by revolving the asteroid $x^{2/3} + y^{2/3} = a^{2/3}$ about x-axis.

Solution:

Step 1: Use parametric form: $x = a\cos^3 t, y = a\sin^3 t, 0 \leq t \leq \pi$ for upper half.

Step 2: Compute derivatives.

$$\frac{dx}{dt} = -3a\cos^2 t\sin t, \frac{dy}{dt} = 3a\sin^2 t\cos t$$

Step 3: Surface area formula.

$$S = 2 \cdot 2\pi \int_0^{\pi/2} y\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} dt$$

$$= 4\pi \int_0^{\pi/2} a\sin^3 t \cdot \sqrt{9a^2\cos^4 t\sin^2 t + 9a^2\sin^4 t\cos^2 t} dt$$

$$= 4\pi \int_0^{\pi/2} a\sin^3 t \cdot 3a \mid \cos t\sin t \mid dt$$

$$= 12\pi a^2 \int_0^{\pi/2} \sin^4 t\cos t dt$$

Step 4: Integrate.
Let $u = \sin t, du = \cos t dt$:

$$12\pi a^2 \int_0^1 u^4 du = 12\pi a^2 [\frac{u^5}{5}]_0^1 = \frac{12\pi a^2}{5}$$

Final Answer:

$$\boxed{\frac{12\pi a^2}{5}}$$