

Assignment – 4

Topic: Taylor and Maclaurin series

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Question 1: Express the following functions in power series using the formula of Maclaurin series:

- (1) e^x
- (2) $\sin x$
- (3) $\log(1+x)$
- (4) $\tan^{-1} x$

Solution:

Part (1): e^x

Step 1: Find derivatives.

$$f(x) = e^x, f'(x) = e^x, f''(x) = e^x, f^{(n)}(x) = e^x$$

$$f(0) = 1, f'(0) = 1, f''(0) = 1, f^{(n)}(0) = 1$$

Step 2: Apply Maclaurin series formula.

$$e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

Part (2): $\sin x$

Step 1: Find derivatives.

$$f(x) = \sin x, f'(x) = \cos x, f''(x) = -\sin x, f'''(x) = -\cos x, f^{(4)}(x) = \sin x$$

$$f(0) = 0, f'(0) = 1, f''(0) = 0, f'''(0) = -1, f^{(4)}(0) = 0$$

Step 2: Apply Maclaurin series formula.

$$\sin x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = 0 + x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} - \dots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

Part (3): $\log(1+x)$

Step 1: Find derivatives.

$$f(x) = \log(1+x), f'(x) = \frac{1}{1+x}, f''(x) = -\frac{1}{(1+x)^2}, f'''(x) = \frac{2}{(1+x)^3}$$

$$f(0) = 0, f'(0) = 1, f''(0) = -1, f'''(0) = 2, f^{(4)}(0) = -6$$

Step 2: Apply Maclaurin series formula.

$$\log(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

Part (4): $\tan^{-1} x$

Step 1: Find derivatives.

$$f(x) = \tan^{-1} x, f'(x) = \frac{1}{1+x^2}, f''(x) = -\frac{2x}{(1+x^2)^2}, \dots$$

$$f(0) = 0, f'(0) = 1, f''(0) = 0, f'''(0) = -2, f^{(4)}(0) = 0, f^{(5)}(0) = 24$$

Step 2: Apply Maclaurin series formula.

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

Question 2: Expand the following functions in powers of $(x-a)$ using Taylor's series.

(1) $f(x) = x^3 - 2x + 4, a = 2$

$$(2) f(x) = x^4 - 11x^3 + 43x^2 - 60x + 14, a = 3$$

Solution:

Part (1): $f(x) = x^3 - 2x + 4, a = 2$

Step 1: Compute derivatives.

$$f'(x) = 3x^2 - 2, f''(x) = 6x, f'''(x) = 6, f^{(4)}(x) = 0$$

Step 2: Evaluate at $x = 2$.

$$\begin{aligned} f(2) &= 8 - 4 + 4 = 8 \\ f'(2) &= 12 - 2 = 10, f''(2) = 12, f'''(2) = 6 \end{aligned}$$

Step 3: Apply Taylor series.

$$\begin{aligned} f(x) &= f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3 \\ &= 8 + 10(x-2) + 6(x-2)^2 + (x-2)^3 \\ &\boxed{f(x) = 8 + 10(x-2) + 6(x-2)^2 + (x-2)^3} \end{aligned}$$

Part (2): $f(x) = x^4 - 11x^3 + 43x^2 - 60x + 14, a = 3$

Step 1: Compute derivatives.

$$\begin{aligned} f'(x) &= 4x^3 - 33x^2 + 86x - 60 \\ f''(x) &= 12x^2 - 66x + 86 \\ f'''(x) &= 24x - 66 \\ f^{(4)}(x) &= 24 \end{aligned}$$

Step 2: Evaluate at $x = 3$.

$$\begin{aligned} f(3) &= 81 - 297 + 387 - 180 + 14 = 5 \\ f'(3) &= 108 - 297 + 258 - 60 = 9 \\ f''(3) &= 108 - 198 + 86 = -4 \\ f'''(3) &= 72 - 66 = 6 \\ f^{(4)}(3) &= 24 \end{aligned}$$

Step 3: Apply Taylor series.

$$\begin{aligned}
 f(x) &= 5 + 9(x-3) - \frac{4}{2!}(x-3)^2 + \frac{6}{3!}(x-3)^3 + \frac{24}{4!}(x-3)^4 \\
 &= 5 + 9(x-3) - 2(x-3)^2 + (x-3)^3 + (x-3)^4 \\
 \boxed{f(x) &= 5 + 9(x-3) - 2(x-3)^2 + (x-3)^3 + (x-3)^4}
 \end{aligned}$$

Question 3: Expand $\sin\left(\frac{\pi}{4} + x\right)$ in powers of x . Find approximate value of $\sin 46^\circ$ and $\sin 44^\circ$.

Solution:

Step 1: Let $f(t) = \sin t$, $t = \frac{\pi}{4} + x$, expand about $\frac{\pi}{4}$.

Step 2: Compute derivatives at $\frac{\pi}{4}$.

$$\begin{aligned}
 f\left(\frac{\pi}{4}\right) &= \frac{\sqrt{2}}{2}, f'\left(\frac{\pi}{4}\right) = \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2} \\
 f''\left(\frac{\pi}{4}\right) &= -\sin \frac{\pi}{4} = -\frac{\sqrt{2}}{2}, f'''\left(\frac{\pi}{4}\right) = -\cos \frac{\pi}{4} = -\frac{\sqrt{2}}{2}
 \end{aligned}$$

Step 3: Taylor expansion.

$$\begin{aligned}
 \sin\left(\frac{\pi}{4} + x\right) &= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{4}x^2 - \frac{\sqrt{2}}{12}x^3 + \dots \\
 \boxed{\sin\left(\frac{\pi}{4} + x\right) &= \frac{\sqrt{2}}{2}\left(1 + x - \frac{x^2}{2} - \frac{x^3}{6} + \dots\right)}
 \end{aligned}$$

Step 4: For $\sin 46^\circ$, $46^\circ = \frac{\pi}{4} + 1^\circ = \frac{\pi}{4} + \frac{\pi}{180}$, so $x = \frac{\pi}{180} \approx 0.017453$.

$$\begin{aligned}
 \sin 46^\circ &\approx \frac{\sqrt{2}}{2}\left(1 + 0.017453 - \frac{(0.017453)^2}{2}\right) \\
 &\approx 0.707107 \times (1.017453 - 0.000152) \approx 0.707107 \times 1.017301 \approx 0.71934 \\
 \boxed{\sin 46^\circ &\approx 0.71934}
 \end{aligned}$$

Step 5: For $\sin 44^\circ$, $x = -\frac{\pi}{180} \approx -0.017453$.

$$\begin{aligned}
 \sin 44^\circ &\approx \frac{\sqrt{2}}{2}\left(1 - 0.017453 - \frac{(0.017453)^2}{2}\right) \\
 &\approx 0.707107 \times (0.982547 - 0.000152) \approx 0.707107 \times 0.982395 \approx 0.69466 \\
 \boxed{\sin 44^\circ &\approx 0.69466}
 \end{aligned}$$

Question 4: Using Taylor's series find approximate value of $\sqrt{36.12}$ and $\sqrt{9.12}$.

Solution:

Part (1): $\sqrt{36.12}$

Step 1: Let $f(x) = \sqrt{x}$, expand about $a = 36$, $x = 36.12$, so $h = 0.12$.

Step 2: Compute derivatives.

$$f(36) = 6, f'(x) = \frac{1}{2\sqrt{x}} \Rightarrow f'(36) = \frac{1}{12}$$

$$f''(x) = -\frac{1}{4x^{3/2}} \Rightarrow f''(36) = -\frac{1}{4 \cdot 216} = -\frac{1}{864}$$

Step 3: Taylor expansion.

$$\begin{aligned} \sqrt{36.12} &\approx 6 + \frac{0.12}{12} - \frac{1}{2} \cdot \frac{1}{864} \cdot (0.12)^2 \\ &= 6 + 0.01 - \frac{1}{1728} \cdot 0.0144 \\ &= 6.01 - 0.00000833 \approx 6.00999167 \\ \boxed{\sqrt{36.12} \approx 6.00999} \end{aligned}$$

Part (2): $\sqrt{9.12}$

Step 1: Expand $f(x) = \sqrt{x}$ about $a = 9$, $h = 0.12$.

Step 2: Compute derivatives.

$$f(9) = 3, f'(9) = \frac{1}{6}, f''(9) = -\frac{1}{4 \cdot 27} = -\frac{1}{108}$$

Step 3: Taylor expansion.

$$\begin{aligned} \sqrt{9.12} &\approx 3 + \frac{0.12}{6} - \frac{1}{2} \cdot \frac{1}{108} \cdot (0.12)^2 \\ &= 3 + 0.02 - \frac{1}{216} \cdot 0.0144 \\ &= 3.02 - 0.00006667 \approx 3.01993333 \end{aligned}$$

$$\sqrt{9.12} \approx 3.01993$$

Question 5: Find absolute extreme value of the function $f(x) = x + \ln x$; $0.5 < x < 4$.

Solution:

Step 1: Find derivative.

$$f'(x) = 1 + \frac{1}{x}$$

Step 2: Set derivative to zero.

$$1 + \frac{1}{x} = 0 \Rightarrow \frac{1}{x} = -1 \Rightarrow x = -1$$

Not in domain $(0.5, 4)$, so no critical point inside.

Step 3: Evaluate at endpoints.

$$\begin{aligned} f(0.5) &= 0.5 + \ln 0.5 \approx 0.5 - 0.6931 = -0.1931 \\ f(4) &= 4 + \ln 4 \approx 4 + 1.3863 = 5.3863 \end{aligned}$$

Step 4: Compare values.

Absolute minimum: -0.1931 at $x = 0.5$

Absolute maximum: 5.3863 at $x = 4$

$$\boxed{\text{Abs min: } -0.1931 \text{ at } x = 0.5, \text{ Abs max: } 5.3863 \text{ at } x = 4}$$

Question 6: Find the extreme values (absolute & local) of the function $y = \frac{x}{x^2+1}$ over its natural domain.

Solution:

Step 1: Domain: all real x .

Step 2: Find derivative.

$$y' = \frac{(x^2 + 1)(1) - x(2x)}{(x^2 + 1)^2} = \frac{x^2 + 1 - 2x^2}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2}$$

Step 3: Set $y' = 0$.

$$1 - x^2 = 0 \Rightarrow x = \pm 1$$

Step 4: Evaluate y at these points.

$$y(1) = \frac{1}{1+1} = \frac{1}{2}$$

$$y(-1) = \frac{-1}{1+1} = -\frac{1}{2}$$

Step 5: Check limits as $x \rightarrow \pm\infty$.

$$\lim_{x \rightarrow \pm\infty} \frac{x}{x^2 + 1} = 0$$

Step 6: Determine nature.

As $x \rightarrow \pm\infty$, $y \rightarrow 0$.

At $x = 1$, $y = 0.5$ is a local maximum.

At $x = -1$, $y = -0.5$ is a local minimum.

No absolute max/min since domain is all real and function approaches but does not attain values greater than 0.5 or less than -0.5 except at these finite points.

Local max: $\frac{1}{2}$ at $x = 1$, Local min: $-\frac{1}{2}$ at $x = -1$
