Assignment – 5

Topic: Indeterminate forms and L'Hospital's rule,

(**Disclaimer:** The purpose of these AI-generated responses is just education and reference. Utilise them to grasp topics and structure, but always rewrite in your own words and double-check the content before submitting. Academic misuse is not the creator's fault.)

Question 1: State L'Hospital's Rule and use it to evaluate the following limits.

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(1) $\lim_{x\to 0} \frac{1-\cos x}{x^2}$

Step 1: Check form: $\frac{0}{0}$.

Step 2: Apply L'Hospital's Rule.

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{\sin x}{2x}$$

Step 3: Still $\frac{0}{0}$, apply again.

$$= \lim_{x\to 0} \frac{\cos x}{2} = \frac{1}{2}$$

$$\boxed{\frac{1}{2}}$$

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(2) $\lim_{x\to 0} \frac{xe^x - \log(1+x)}{x^2}$

Step 1: Check form: $\frac{0}{0}$.

Step 2: Apply L'Hospital's Rule.

$$\lim_{x\to 0} \frac{e^x + xe^x - \frac{1}{1+x}}{2x}$$

At $x = 0$: numerator = $1 + 0 - 1 = 0$, still $\frac{0}{0}$.

Step 3: Differentiate numerator and denominator again.
Numerator derivative: $e^x + e^x + xe^x + \frac{1}{(1+x)^2} = 2e^x + xe^x + \frac{1}{(1+x)^2}$
At $x = 0$: $2 + 0 + 1 = 3$
Denominator derivative: 2

So limit $= \frac{3}{2}$

$$\boxed{\frac{3}{2}}$$

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(3) $\lim_{x\to 0} \frac{e^x - e^{-x} - 2\log(1+x)}{x\sin x}$

Step 1: Check form: $\frac{0}{0}$.

Step 2: Apply L'Hospital's Rule.
Numerator derivative: $e^x + e^{-x} - \frac{2}{1+x}$
At $x = 0$: $1 + 1 - 2 = 0$
Denominator derivative: $\sin x + x\cos x$, at $x = 0$: $0 + 0 = 0$, still $\frac{0}{0}$.

Step 3: Differentiate again.
Numerator: $e^x - e^{-x} + \frac{2}{(1+x)^2}$
At $x = 0$: $1 - 1 + 2 = 2$
Denominator: $\cos x + \cos x - x\sin x = 2\cos x - x\sin x$
At $x = 0$: $2 - 0 = 2$

So limit $= \frac{2}{2} = 1$

$$\boxed{1}$$

Question 2: Determine indeterminate forms and evaluate the limit.

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(1) $\lim_{x\to 0}\left[\frac{1}{x^2} - \frac{1}{\sin^2 x}\right]$

Step 1: Common denominator: $\frac{\sin^2 x - x^2}{x^2\sin^2 x}$, form $\frac{0}{0}$.

Step 2: Apply L'Hospital's Rule.
Derivative of numerator: $2\sin x\cos x - 2x = \sin 2x - 2x$
Derivative of denominator: $2x\sin^2 x + x^2 \cdot 2\sin x\cos x = 2x\sin^2 x + x^2\sin 2x$
Still $\frac{0}{0}$ at $x = 0$.

Step 3: Apply L'Hospital's Rule again.
Numerator: $2\cos 2x - 2$

Denominator: $2\sin^2 x + 4x\sin x\cos x + 2x\sin 2x + x^2 \cdot 2\cos 2x$
At $x = 0$: numerator = $2 - 2 = 0$, denominator = $0 + 0 + 0 + 0 = 0$, still $\frac{0}{0}$.

Step 4: Differentiate again.
Numerator: $-4\sin 2x$
Denominator: $4\sin x\cos x + 4\sin x\cos x + 4x\cos^2 x - 4x\sin^2 x + 2\sin 2x + 4x\cos 2x + 4x\cos 2x - 4x^2\sin 2x$
Simplify: $8\sin x\cos x + 4x(\cos^2 x - \sin^2 x) + 2\sin 2x + 8x\cos 2x - 4x^2\sin 2x$
At $x = 0$: $0 + 0 + 0 + 0 - 0 = 0$, still $\frac{0}{0}$.

Step 5: Differentiate again.
Numerator: $-8\cos 2x$
At $x = 0$: -8
Denominator derivative (from last step):
Term1: $8\cos 2x$
Term2: $4(\cos^2 x - \sin^2 x) + 4x(-2\cos x\sin x - 2\sin x\cos x) = 4\cos 2x - 16x\sin x\cos x$
Term3: $4\cos 2x$
Term4: $8\cos 2x - 16x\sin 2x$
Term5: $-8x\sin 2x - 8x^2\cos 2x$
Sum at $x = 0$: $8 + 4 + 4 + 8 + 0 + 0 = 24$

So limit = $\frac{-8}{24} = -\frac{1}{3}$

$$\boxed{-\frac{1}{3}}$$

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(2) $\lim_{x \to 0} (\frac{1^x + 2^x + 3^x}{3})^{\frac{1}{x}}$

Step 1: Let $L = \lim_{x \to 0} (\frac{1 + 2^x + 3^x}{3})^{\frac{1}{x}}$

Step 2: Take natural log.

$$\ln L = \lim_{x \to 0} \frac{\ln(\frac{1 + 2^x + 3^x}{3})}{x}$$

Form $\frac{0}{0}$.

Step 3: Apply L'Hospital's Rule.

$$\ln\ L = \lim_{x \to 0} \frac{\frac{2^x \ln\ 2 + 3^x \ln\ 3}{1 + 2^x + 3^x}}{1}$$

At $x = 0$: numerator = $\ln\ 2 + \ln\ 3$, denominator = 3
So $\ln\ L = \frac{\ln\ 2 + \ln\ 3}{3} = \frac{\ln 6}{3}$

Thus $L = e^{(\ln\ 6)/3} = 6^{1/3}$

$$\boxed{\sqrt[3]{6}}$$

Question 3: $\lim_{x \to 0} (\frac{1}{x})^{\tan x}$

Step 1: Let $L = \lim_{x \to 0} (\frac{1}{x})^{\tan x}$

Step 2: Take natural log.

$$\ln\ L = \lim_{x \to 0} \tan\ x \cdot \ln\left(\frac{1}{x}\right) = \lim_{x \to 0} \frac{-\ln\ x}{\cot x}$$

Form $\frac{\infty}{\infty}$.

Step 3: Apply L'Hospital's Rule.

$$\ln\ L = \lim_{x \to 0} \frac{-1/x}{-\csc^2 x} = \lim_{x \to 0} \frac{\sin^2 x}{x}$$
$$= \lim_{x \to 0} \frac{\sin x}{x} \cdot \sin\ x = 1 \cdot 0 = 0$$

So $L = e^0 = 1$

$$\boxed{1}$$

Question 4: Evaluate $\lim_{x \to \sqrt{2}} \frac{\sec x}{1 + \tan\ x}$

Step 1: Direct substitution: $x = \sqrt{2}$ radians.

$$\sec(\sqrt{2}) = \frac{1}{\cos(\sqrt{2})}, \tan(\sqrt{2}) = \frac{\sin(\sqrt{2})}{\cos(\sqrt{2})}$$

Denominator: $1 + \frac{\sin(\sqrt{2})}{\cos(\sqrt{2})} = \frac{\cos(\sqrt{2})+\sin(\sqrt{2})}{\cos(\sqrt{2})}$

So expression $= \frac{1/\cos(\sqrt{2})}{(\cos(\sqrt{2})+\sin(\sqrt{2}))/\cos(\sqrt{2})} = \frac{1}{\cos(\sqrt{2})+\sin(\sqrt{2})}$

No indeterminate form, just substitute $x = \sqrt{2}$:

$$\boxed{\frac{1}{\cos(\sqrt{2}) + \sin(\sqrt{2})}}$$

Question 5: Evaluate $\lim_{x\to 0+} \sqrt{x\ln x}$

Step 1: Let $L = \lim_{x\to 0+} \sqrt{x\ln x}$

Step 2: Consider $\ln L = \lim_{x\to 0+} \frac{1}{2}\ln(x\ln x) = \frac{1}{2}\lim_{x\to 0+}[\ln x + \ln(\ln x)]$

As $x \to 0+$, $\ln x \to -\infty$, $\ln(\ln x) \to \ln(-\infty)$ undefined real? Actually $\ln x \to -\infty$, so $\ln(\ln x)$ not real for $\ln x < 0$. Better approach:

Let $t = -\ln x$, then $x = e^{-t}$, as $x \to 0+$, $t \to +\infty$.

Then $x\ln x = e^{-t} \cdot (-t) = -te^{-t}$

So $\sqrt{x\ln x} = \sqrt{-te^{-t}}$ — not real for large t since inside negative. So limit does not exist in reals.

But maybe they mean: $\lim_{x\to 0+} \sqrt{x}\ln x$ instead? But given is $\sqrt{x\ln x}$. Inside $x\ln x \to 0^-$, so square root of negative number → not real. So:

$$\boxed{\text{Does not exist (not real)}}$$