Assignment – 6

Topic: Infinite Sequence & Series Infinite series

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Question 1: Explain Convergence and Divergence of sequence. Examine whether the following sequences are convergent or divergent.
(1) $\{\frac{1}{n}\}_{n=1}^{\infty}$
(2) $\{(-1)^{n+1}\}_{n=1}^{\infty}$

Solution:

(1) $\{\frac{1}{n}\}$

Step 1: Limit as $n \to \infty$:

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

Since the limit exists and is finite, the sequence converges.

$$\boxed{\text{Converges}}$$

(2) $\{(-1)^{n+1}\}$

Step 1: Terms oscillate: $1, -1, 1, -1, \ldots$

Step 2: No single limit, so sequence diverges.

$$\boxed{\text{Diverges}}$$

Question 2: Describe the sandwich theorem for sequences. Use it to check convergence of following sequences.
(1) $\{\frac{\cos n}{n}\}_{n=1}^{\infty}$
(2) $\{\frac{(-1)^{n+1}}{2n-1}\}_{n=1}^{\infty}$

Solution:

(1) $\{\frac{\cos n}{n}\}$

Step 1: Bound the sequence:

$$-1 \leq \cos\ n \leq 1 \Rightarrow -\frac{1}{n} \leq \frac{\cos n}{n} \leq \frac{1}{n}$$

Step 2: As $n \to \infty$, $-\frac{1}{n} \to 0$ and $\frac{1}{n} \to 0$.
By Sandwich Theorem, $\frac{\cos n}{n} \to 0$.
So sequence converges.

$$\boxed{\text{Converges}}$$

(2) $\{\frac{(-1)^{n+1}}{2n-1}\}$

Step 1: Bound the sequence:

$$-\frac{1}{2n-1} \leq \frac{(-1)^{n+1}}{2n-1} \leq \frac{1}{2n-1}$$

Step 2: As $n \to \infty$, both bounds $\to 0$.
By Sandwich Theorem, sequence $\to 0$.
So converges.

$$\boxed{\text{Converges}}$$

Question 3: Write down continuous function theorem for sequence. Use it to examine convergence of following sequences.
(1) $\{\frac{1}{2^n}\}_{n=1}^{\infty}$

Solution:

Step 1: Limit:

$$\lim_{n\to\infty} \frac{1}{2^n} = 0$$

Since the limit exists and is finite, sequence converges.

$$\boxed{\text{Converges}}$$

Question 4: Explain telescoping series and use it to check convergence of following series.

(1) $\sum_{n=1}^{\infty} \frac{4}{(4n-3)(4n+1)}$

(2) $\sum_{n=1}^{\infty} (\tan^{-1}(n) - \tan^{-1}(n+1))$

Solution:

(1) $\sum_{n=1}^{\infty} \frac{4}{(4n-3)(4n+1)}$

Step 1: Partial fractions:

$$\frac{4}{(4n-3)(4n+1)} = \frac{A}{4n-3} + \frac{B}{4n+1}$$
$$4 = A(4n+1) + B(4n-3)$$
$$4n: 4A + 4B = 0 \Rightarrow A + B = 0$$
$$\text{Constant: } A - 3B = 4$$

Solve: $A = 1, B = -1$

So term = $\frac{1}{4n-3} - \frac{1}{4n+1}$

Step 2: Partial sum S_N:

$$S_N = \sum_{n=1}^{N} \left(\frac{1}{4n-3} - \frac{1}{4n+1}\right)$$

Telescopes:

$$n = 1: 1 - \frac{1}{5}$$
$$n = 2: \frac{1}{5} - \frac{1}{9}$$

…

$$n = N: \frac{1}{4N-3} - \frac{1}{4N+1}$$

Sum = $1 - \frac{1}{4N+1}$

Step 3: Limit as $N \to \infty$:

$$\lim_{N\to\infty} S_N = 1 - 0 = 1$$

So series converges to 1.

$$\boxed{\text{Converges}}$$

(2) $\sum_{n=1}^{\infty} \quad (\tan^{-1}(n) - \tan^{-1}(n+1))$

Step 1: Partial sum S_N:

$$S_N = (\tan^{-1} 1 - \tan^{-1} 2) + (\tan^{-1} 2 - \tan^{-1} 3) + \cdots + (\tan^{-1} N - \tan^{-1}(N + 1))$$

Telescopes to: $\tan^{-1} 1 - \tan^{-1}(N + 1)$

Step 2: Limit as $N \to \infty$:

$$\lim_{N\to\infty} S_N = \tan^{-1} 1 - \frac{\pi}{2} = \frac{\pi}{4} - \frac{\pi}{2} = -\frac{\pi}{4}$$

So series converges.

$$\boxed{\text{Converges}}$$

Question 5: Define geometric series and explain its convergence. Examine convergence of following series and find sum of series if it is convergent.
$\sum_{n=1}^{\infty} \quad \frac{(-3)^n}{2^{n+2}}$

Solution:

Step 1: Rewrite series:

$$\sum_{n=1}^{\infty} \frac{(-3)^n}{2^{n+2}} = \frac{1}{4}\sum_{n=1}^{\infty} \quad (\frac{-3}{2})^n$$

Step 2: Geometric series with $r = -\frac{3}{2}$.
Since $| r |= 1.5 > 1$, series diverges.

$$\boxed{\text{Diverges}}$$

Question 6: State n^{th} term test for divergence of an infinite series. Examine that following series are divergent.

(1) $\sum_{n=1}^{\infty} \frac{2n}{3n-1}$
(2) $\sum_{n=1}^{\infty} n\sin\frac{1}{n}$

Solution:

(1) $\sum_{n=1}^{\infty} \frac{2n}{3n-1}$

Step 1: Check n-th term:

$$\lim_{n\to\infty} \frac{2n}{3n-1} = \frac{2}{3} \neq 0$$

So by n-th term test, series diverges.

$$\boxed{\text{Diverges}}$$

(2) $\sum_{n=1}^{\infty} n\sin\frac{1}{n}$

Step 1: Check n-th term:

$$\lim_{n\to\infty} n\sin\frac{1}{n} = \lim_{t\to 0} \frac{\sin t}{t} = 1 \neq 0$$

So by n-th term test, series diverges.

$$\boxed{\text{Diverges}}$$

Question 7: Examine following series for their convergence using Integral test.
(1) $\sum_{n=1}^{\infty} \frac{\tan^{-1}(n)}{1+n^2}$
(2) $\sum_{n=1}^{\infty} \frac{e^n}{1+e^{2n}}$

Solution:

(1) $\sum_{n=1}^{\infty} \frac{\tan^{-1}(n)}{1+n^2}$

Step 1: Consider $f(x) = \frac{\tan^{-1} x}{1+x^2}$, continuous, positive, decreasing for $x \geq 1$.

Step 2: Evaluate $\int_1^\infty f(x)dx$:
Let $u = \tan^{-1} x$, $du = \frac{1}{1+x^2}dx$.
When $x = 1$, $u = \frac{\pi}{4}$; when $x \to \infty$, $u \to \frac{\pi}{2}$.

$$\int_1^\infty \frac{\tan^{-1} x}{1+x^2}dx = \int_{\pi/4}^{\pi/2} udu = [\frac{u^2}{2}]_{\pi/4}^{\pi/2} = \frac{(\pi/2)^2}{2} - \frac{(\pi/4)^2}{2} = \frac{\pi^2}{8} - \frac{\pi^2}{32}$$
$$= \frac{3\pi^2}{32}$$

Finite, so by Integral test, series converges.

$$\boxed{\text{Converges}}$$

(2) $\sum_{n=1}^\infty \frac{e^n}{1+e^{2n}}$

Step 1: $f(x) = \frac{e^x}{1+e^{2x}}$, continuous, positive, decreasing.

Step 2: Evaluate $\int_1^\infty \frac{e^x}{1+e^{2x}}dx$:
Let $u = e^x$, $du = e^x dx$.
When $x = 1$, $u = e$; when $x \to \infty$, $u \to \infty$.

$$\int_1^\infty \frac{e^x}{1+e^{2x}}dx = \int_e^\infty \frac{1}{1+u^2}du = [\tan^{-1} u]_e^\infty = \frac{\pi}{2} - \tan^{-1} e$$

Finite, so by Integral test, series converges.

$$\boxed{\text{Converges}}$$