Assignment – 7

Topic: Infinite Sequence & Series Infinite series

Question 1: Prove that the P-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

Solution:

Step 1: Use the Integral Test. Consider $f(x) = \frac{1}{x^p}$, continuous, positive, decreasing for $x \geq 1$.

Step 2: Evaluate $\int_1^{\infty} x^{-p} dx$:

$$\int_1^{\infty} x^{-p} dx = [\frac{x^{1-p}}{1-p}]_1^{\infty} \text{for } p \neq 1$$

If $p > 1$, $1 - p < 0$, so $x^{1-p} \to 0$ as $x \to \infty$, integral = $\frac{1}{p-1}$ (finite) ⇒ converges.

If $p < 1$, $1 - p > 0$, $x^{1-p} \to \infty$ as $x \to \infty$ ⇒ diverges.

Step 3: For $p = 1$:

$$\int_1^{\infty} \frac{1}{x} dx = [\ln x]_1^{\infty} = \infty \Rightarrow \text{diverges.}$$

Conclusion:

$$\boxed{\text{Converges for } p > 1, \text{ diverges for } p \leq 1}$$

Question 2: Apply comparison test to find convergence of following series.

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(1) $\sum_{n=1}^{\infty} \frac{1}{n^2+30}$

Step 1: Compare with $\sum \frac{1}{n^2}$ (convergent p-series, $p = 2$):

$$\frac{1}{n^2 + 30} < \frac{1}{n^2}$$

Since larger series converges, given series converges by Comparison Test.

$$\boxed{\text{Converges}}$$

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(2) $\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^{3/2}}$

Step 1: $0 \le \cos^2 n \le 1$, so

$$0 \le \frac{\cos^2 n}{n^{3/2}} \le \frac{1}{n^{3/2}}$$

$\sum \frac{1}{n^{3/2}}$ converges (p-series, $p = 1.5 > 1$), so given series converges.

$$\boxed{\text{Converges}}$$

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(3) $\sum_{n=1}^{\infty} \frac{3n^2-3n}{n^2(n-1)(n^2+5)}$

Step 1: Simplify general term for large n:

$$\frac{3n^2 - 3n}{n^2(n-1)(n^2+5)} \approx \frac{3n^2}{n^2 \cdot n \cdot n^2} = \frac{3}{n^3}$$

Compare with $\sum \frac{1}{n^3}$ (convergent). Limit comparison:

$$\lim_{n\to\infty} \frac{a_n}{1/n^3} = \lim_{n\to\infty} \frac{3n^5 - 3n^4}{n^2(n-1)(n^2+5)} = 3$$

Since limit is finite and positive, both series converge.

$$\boxed{\text{Converges}}$$

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(4) $\sum_{n=1}^{\infty} \frac{1}{n3^n}$

Step 1: Compare with $\sum \frac{1}{3^n}$ (convergent geometric, $r = 1/3$):

$$\frac{1}{n3^n} \le \frac{1}{3^n} \text{ for } n \ge 1$$

So given series converges.

$$\boxed{\text{Converges}}$$

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(5) $\sum_{n=1}^{\infty} \quad (\frac{n}{3n+1})^n$

Step 1: Use Root Test:

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{n}{3n+1} = \frac{1}{3} < 1$$

So series converges.

$$\boxed{\text{Converges}}$$

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(6) $\sum_{n=2}^{\infty} \quad \ln(1 + \frac{1}{n^2})$

Step 1: For large n, $\ln(1 + 1/n^2) \sim \frac{1}{n^2}$.
Compare with $\sum 1/n^2$ (convergent). Limit comparison:

$$\lim_{n\to\infty} \frac{\ln(1 + 1/n^2)}{1/n^2} = 1$$

So both series converge.

$$\boxed{\text{Converges}}$$

Question 3: Examine convergence of following series by using Ratio Test.

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(1) $\sum_{n=1}^{\infty} \quad \frac{(n!)^2}{2^{n^2}}$

Step 1: Ratio Test:

$$\frac{a_{n+1}}{a_n} = \frac{((n+1)!)^2}{2^{(n+1)^2}} \cdot \frac{2^{n^2}}{(n!)^2} = \frac{(n+1)^2}{2^{2n+1}}$$
$$\lim_{n\to\infty} \frac{(n+1)^2}{2^{2n+1}} = 0 < 1$$

So converges.

$$\boxed{\text{Converges}}$$

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(2) $\sum_{n=1}^{\infty} \frac{4^n(n+1)!}{n^{n+1}}$

Step 1: Ratio Test:

$$\frac{a_{n+1}}{a_n} = \frac{4^{n+1}(n+2)!}{(n+1)^{n+2}} \cdot \frac{n^{n+1}}{4^n(n+1)!} = 4 \cdot (n+2) \cdot \frac{n^{n+1}}{(n+1)^{n+2}}$$
$$= 4 \cdot \frac{n+2}{n+1} \cdot (\frac{n}{n+1})^{n+1}$$
$$\lim_{n\to\infty} (\frac{n}{n+1})^{n+1} = e^{-1}, \lim_{n\to\infty} \frac{n+2}{n+1} = 1$$

So limit = $4 \cdot 1 \cdot e^{-1} = \frac{4}{e} \approx 1.47 > 1 \Rightarrow$ diverges.

$$\boxed{\text{Diverges}}$$

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(3) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

Step 1: Ratio Test:

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \frac{n+1}{(n+1)^{n+1}} \cdot n^n = \frac{n^n}{(n+1)^n}$$
$$= (\frac{n}{n+1})^n = \frac{1}{(1+\frac{1}{n})^n} \to \frac{1}{e} \approx 0.367 < 1$$

So converges.

$$\boxed{\text{Converges}}$$

Question 4: Examine convergence of following series by using Cauchy Root Test.

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(1) $\sum_{n=1}^{\infty} \quad ne^{-n^2}$

Step 1: Root Test:

$$\sqrt[n]{a_n} = \sqrt[n]{n} \cdot e^{-n}$$
$$\lim_{n\to\infty} \sqrt[n]{n} = 1, \lim_{n\to\infty} e^{-n} = 0$$

So limit = 0 < 1 ⇒ converges.

$$\boxed{\text{Converges}}$$

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(2) $\sum_{n=1}^{\infty} \quad (1+\frac{1}{n})^{n^2}$

Step 1: Root Test:

$$\sqrt[n]{a_n} = (1+\frac{1}{n})^n \to e > 1$$

So diverges.

$$\boxed{\text{Diverges}}$$

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(3) $\sum_{n=2}^{\infty} \quad \frac{n}{(\log\ n)^n}$

Step 1: Root Test:

$$\sqrt[n]{a_n} = \frac{\sqrt[n]{n}}{\log n}$$
$$\lim_{n\to\infty} \sqrt[n]{n} = 1, \lim_{n\to\infty} \log\ n = \infty$$

So limit = 0 < 1 ⇒ converges.

$$\boxed{\text{Converges}}$$

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