Assignment – 9

Topic: Partial Derivative

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Question 1: Find the domain and range and sketch the graph of domain of following function

(1) $f(x, y) = \sqrt{x^2 - y}$

(2) $f(x, y) = \frac{x^2}{\sqrt{x^2+y^2-4}}$

Solution:

(1) $f(x, y) = \sqrt{x^2 - y}$

Step 1: Domain: $x^2 - y \geq 0 \Rightarrow y \leq x^2$

Step 2: Range: $\sqrt{x^2 - y} \geq 0 \Rightarrow [0, \infty)$

$$\boxed{\text{Domain: } y \leq x^2, \text{ Range: } [0, \infty)}$$

(2) $f(x, y) = \frac{x^2}{\sqrt{x^2+y^2-4}}$

Step 1: Domain: $x^2 + y^2 - 4 > 0 \Rightarrow$ outside circle $x^2 + y^2 = 4$ (not including boundary).

Step 2: Range: $x^2 \geq 0$, denominator > 0, so $f \geq 0$. Can be arbitrarily large near boundary or as $|x|$ large $\Rightarrow (0, \infty)$.

$$\boxed{\text{Domain: } x^2 + y^2 > 4, \text{ Range: } (0, \infty)}$$

Question 2: Sketch the following surfaces by using level curves

(1) $z = x^2 + y^2$

(2) $z^2 = x^2 + y^2$

(3) $z = x + y$

Solution:

(1) $z = x^2 + y^2$

Level curves: $x^2 + y^2 = c$ (circles), $c \geq 0$.
Paraboloid.

(2) $z^2 = x^2 + y^2$

Level curves: $x^2 + y^2 = c^2$ (circles), $z = \pm c$.
Double cone.

(3) $z = x + y$

Level curves: $x + y = c$ (straight lines).
Plane.

$$\boxed{\text{(1) Paraboloid, (2) Double cone, (3) Plane}}$$

Question 3: Evaluate following limits if exist.

(1) $\lim_{(x,y)\to(0,0)} \frac{2x^2-2xy}{x-y^2}$

(2) $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$

(3) $\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2}$

Solution:

(1) $\lim_{(x,y)\to(0,0)} \frac{2x^2-2xy}{x-y^2}$

Step 1: Approach along $y = 0$: $\frac{2x^2}{x} = 2x \to 0$.
Approach along $x = 0$: $0/(-y^2) = 0$.
Approach along $y = x$: $\frac{2x^2-2x^2}{x-x^2} = 0/(x - x^2) \to 0$.
Approach along $x = y^2$: $\frac{2y^4-2y^3}{y^2-y^2}$ undefined.
But path $x = y^2 + y^3$:
Numerator: $2(y^2 + y^3)^2 - 2(y^2 + y^3)y = 2y^4 + 4y^5 + 2y^6 - 2y^3 - 2y^4 = -2y^3 + 4y^5 + 2y^6$
Denominator: $y^2 + y^3 - y^2 = y^3$
Ratio: $(-2y^3 + \cdots)/y^3 \to -2$.
Different limits $\Rightarrow$ limit does not exist.

$$\boxed{\text{DNE}}$$

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(2) $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$

Step 1: Approach along $y = 0$: $x^2/x^2 = 1$.
Approach along $x = 0$: $-y^2/y^2 = -1$.
Different limits ⇒ DNE.

$$\boxed{\text{DNE}}$$

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(3) $\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2}$

Step 1: Use polar: $x = r\cos\theta, y = r\sin\theta$:

$$\frac{4r\cos\theta \cdot r^2\sin^2\theta}{r^2} = 4r\cos\theta\sin^2\theta$$

As $r \to 0$, limit = 0.

$$\boxed{0}$$

Question 4: Discuss the continuity of $f(x,y) = \frac{xy}{x^2+y^2}$ if $(x,y) \neq (0,0)$, $f(0,0) = 0$.

Solution:

Step 1: Check limit at (0,0):
Along $y = mx$: $f = \frac{mx^2}{(1+m^2)x^2} = \frac{m}{1+m^2}$, depends on m ⇒ limit DNE.
So not continuous at (0,0).

$$\boxed{\text{Not continuous at (0,0)}}$$

Question 5: If

$$f(x,y) = \frac{2xy(x^2+y^2)}{x^2+y^2} \text{ if } (x,y) \neq (0,0), f(0,0) = 0$$

find $\left(\frac{\partial^2 f}{\partial x\,\partial y}\right)_{(0,0)}$ and $\left(\frac{\partial^2 f}{\partial y\,\partial x}\right)_{(0,0)}$

Solution:

Step 1: Simplify: $f(x,y) = 2xy$ for $(x,y) \neq (0,0)$, and 0 at (0,0).

Step 2: First partials at (0,0):

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = 0$$
$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k) - 0}{k} = 0$$

Step 3: Mixed partials:

$$f_{xy}(0,0) = \lim_{k\to 0} \frac{f_x(0,k) - f_x(0,0)}{k}$$

For $(x,y) \neq (0,0)$, $f_x = 2y$, so $f_x(0,k) = 2k$.
So $f_{xy}(0,0) = \lim_{k\to 0} \frac{2k-0}{k} = 2$.

$$f_{yx}(0,0) = \lim_{h\to 0} \frac{f_y(h,0) - f_y(0,0)}{h}$$

For $(x,y) \neq (0,0)$, $f_y = 2x$, so $f_y(h,0) = 2h$.
So $f_{yx}(0,0) = \lim_{h\to 0} \frac{2h-0}{h} = 2$.

$$\boxed{2, 2}$$

Question 6: For following function show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

(1) $u = \tan^{-1}(\frac{y}{x})$

(2) $u = \log \sqrt{x^2 + y^2}$

Solution:

(1) $u = \tan^{-1}(y/x)$

Step 1: Compute $u_x = \frac{1}{1+(y/x)^2} \cdot (-\frac{y}{x^2}) = -\frac{y}{x^2+y^2}$

$$u_y = \frac{1}{1+(y/x)^2} \cdot \frac{1}{x} = \frac{x}{x^2+y^2}$$

Step 2: Second derivatives:

$$u_{xx} = \frac{2xy}{(x^2+y^2)^2}$$
$$u_{yy} = -\frac{2xy}{(x^2+y^2)^2}$$

Sum = 0.

Shown

(2) $u = \log\sqrt{x^2+y^2} = \frac{1}{2}\log(x^2+y^2)$

Step 1: $u_x = \frac{x}{x^2+y^2}$, $u_y = \frac{y}{x^2+y^2}$

Step 2: $u_{xx} = \frac{(x^2+y^2)-x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$

$$u_{yy} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

Sum = 0.

Shown

Question 7: If $\frac{x^2}{a^2+v}+\frac{y^2}{b^2+v}+\frac{z^2}{c^2+v}=1$ where v is function of x, y, z then prove that

$$(\frac{\partial v}{\partial x})^2+(\frac{\partial v}{\partial y})^2+(\frac{\partial v}{\partial z})^2 = 2(x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}+z\frac{\partial v}{\partial z})$$

Solution:

Step 1: Differentiate w.r.t x:

$$\frac{2x}{a^2+v}-\frac{x^2}{(a^2+v)^2}v_x-\frac{y^2}{(b^2+v)^2}v_x-\frac{z^2}{(c^2+v)^2}v_x=0$$

Factor v_x:

$$\frac{2x}{a^2+v}-v_x[\frac{x^2}{(a^2+v)^2}+\frac{y^2}{(b^2+v)^2}+\frac{z^2}{(c^2+v)^2}]=0$$

Let $K=\sum\frac{x^2}{(a^2+v)^2}$, then $v_x=\frac{2x}{a^2+v}\cdot\frac{1}{K}$.

Similarly $v_y=\frac{2y}{b^2+v}\cdot\frac{1}{K}$, $v_z=\frac{2z}{c^2+v}\cdot\frac{1}{K}$.

Step 2: Compute LHS:

$v_x^2+v_y^2+v_z^2=\frac{4}{K^2}[\frac{x^2}{(a^2+v)^2}+\frac{y^2}{(b^2+v)^2}+\frac{z^2}{(c^2+v)^2}]=\frac{4}{K^2}\cdot K=\frac{4}{K}$.

Step 3: Compute RHS:

$$2(xv_x+yv_y+zv_z)=2[\frac{2x^2}{K(a^2+v)}+\frac{2y^2}{K(b^2+v)}+\frac{2z^2}{K(c^2+v)}]$$

From original equation: $\sum\frac{x^2}{a^2+v}=1$, so RHS = $\frac{4}{K}\cdot 1=\frac{4}{K}$.

LHS = RHS.

$\boxed{\text{Shown}}$

Question 8: If $x = r\cos\ \theta$ and $y = r\sin\ \theta$ prove that

$$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{1}{r}[(\frac{\partial r}{\partial x})^2 + (\frac{\partial r}{\partial y})^2]$$

Solution:

Step 1: $r = \sqrt{x^2 + y^2}$, so $r_x = x/r$, $r_y = y/r$.

Step 2: $r_{xx} = \frac{r - x(x/r)}{r^2} = \frac{r^2 - x^2}{r^3} = \frac{y^2}{r^3}$

$$r_{yy} = \frac{x^2}{r^3}$$

So $r_{xx} + r_{yy} = \frac{x^2+y^2}{r^3} = \frac{r^2}{r^3} = \frac{1}{r}$.

Step 3: RHS: $\frac{1}{r}[\frac{x^2}{r^2} + \frac{y^2}{r^2}] = \frac{1}{r} \cdot \frac{r^2}{r^2} = \frac{1}{r}$.

LHS = RHS.

$\boxed{\text{Shown}}$