Assignment – 10

Topic: Partial Derivative

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Question 1: Explain Chain Rule for composite functions.

Solution:

If $z = f(x, y)$ and $x = g(t), y = h(t)$, then $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$

Question 2: Find $\frac{du}{dt}$ if $u = x^2 + y^2 + z^2, x = e^{2t}, y = e^{2t}\cos 3t, z = e^{2t}\sin 3t$

Solution:

Step 1: Compute partials:

$$\frac{\partial u}{\partial x} = 2x, \frac{\partial u}{\partial y} = 2y, \frac{\partial u}{\partial z} = 2z$$

Step 2: Compute derivatives:

$$\frac{dx}{dt} = 2e^{2t}, \frac{dy}{dt} = 2e^{2t}\cos 3t - 3e^{2t}\sin 3t, \frac{dz}{dt} = 2e^{2t}\sin 3t + 3e^{2t}\cos 3t$$

Step 3: Apply chain rule:

$$\frac{du}{dt} = 2x(2e^{2t}) + 2y(2e^{2t}\cos 3t - 3e^{2t}\sin 3t) + 2z(2e^{2t}\sin 3t + 3e^{2t}\cos 3t)$$

Substitute $x = e^{2t}, y = e^{2t}\cos 3t, z = e^{2t}\sin 3t$:

$$= 4e^{4t} + 4e^{4t}\cos^2 3t - 6e^{4t}\cos 3t\sin 3t + 4e^{4t}\sin^2 3t + 6e^{4t}\sin 3t\cos 3t$$

Simplify: $4e^{4t} + 4e^{4t}(\cos^2 3t + \sin^2 3t) = 4e^{4t} + 4e^{4t} = 8e^{4t}$

$$\boxed{8e^{4t}}$$

Question 3: The height of a right circular cone is 15cm and is increasing at 0.4cm/s. The radius of the base is 10cm and is decreasing at 0.6cm/s. Find the rate of change of volume.

Solution:

Step 1: Volume $V = \frac{1}{3}\pi r^2 h$

Given: $h = 15, r = 10, \frac{dh}{dt} = 0.4, \frac{dr}{dt} = -0.6$

Step 2: Differentiate:

$$\frac{dV}{dt} = \frac{1}{3}\pi(2rh\frac{dr}{dt} + r^2\frac{dh}{dt})$$
$$= \frac{1}{3}\pi(2 \cdot 10 \cdot 15 \cdot (-0.6) + 100 \cdot 0.4)$$
$$= \frac{1}{3}\pi(-180 + 40) = \frac{1}{3}\pi(-140) = -\frac{140\pi}{3} \text{ cm}^3/\text{s}$$
$$\boxed{-\frac{140\pi}{3}}$$

Question 4: If $u = f(x - y, y - x, z - x)$ prove $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

Solution:

Step 1: Let $p = x - y, q = y - x, r = z - x$. Then $u = f(p, q, r)$.

Step 2: Compute partials:

$$\frac{\partial u}{\partial x} = f_p \cdot 1 + f_q \cdot (-1) + f_r \cdot (-1) = f_p - f_q - f_r$$
$$\frac{\partial u}{\partial y} = f_p \cdot (-1) + f_q \cdot 1 + f_r \cdot 0 = -f_p + f_q$$
$$\frac{\partial u}{\partial z} = f_p \cdot 0 + f_q \cdot 0 + f_r \cdot 1 = f_r$$

Step 3: Sum: $(f_p - f_q - f_r) + (-f_p + f_q) + (f_r) = 0$

$$\boxed{\text{Shown}}$$

Question 5: Using partial derivative find $\frac{dy}{dx}$ for $xe^y + \sin(xy) + y\ln 2 = 0$ at $(0, \ln 2)$

Solution:

Step 1: Let $F(x, y) = xe^y + \sin(xy) + y\ln 2$.
Then $\frac{dy}{dx} = -\frac{F_x}{F_y}$.

Step 2: Compute:

$$F_x = e^y + y\cos(xy), F_y = xe^y + x\cos(xy) + \ln 2$$

Step 3: At $(0, \ln 2)$:

$$F_x = e^{\ln 2} + \ln 2 \cdot \cos 0 = 2 + \ln 2$$
$$F_y = 0 + 0 + \ln 2 = \ln 2$$
$$\frac{dy}{dx} = -\frac{2 + \ln 2}{\ln 2}$$
$$\boxed{-\frac{2 + \ln 2}{\ln 2}}$$

Question 6: If $yx^y = \sin x$ then find $\frac{dy}{dx}$

Solution:

Step 1: Take natural log: $\ln y + y\ln x = \ln(\sin x)$

Step 2: Differentiate implicitly:

$$\frac{1}{y}\frac{dy}{dx} + \frac{dy}{dx}\ln x + \frac{y}{x} = \frac{\cos x}{\sin x}$$
$$\frac{dy}{dx}(\frac{1}{y} + \ln x) = \cot x - \frac{y}{x}$$
$$\frac{dy}{dx} = \frac{\cot x - \frac{y}{x}}{\frac{1}{y} + \ln x}$$
$$\boxed{\frac{\cot x - \frac{y}{x}}{\frac{1}{y} + \ln x}}$$

Question 7: If $\sin(xyz) + x^3y^2z^2 + \ln(x^2 + y^2 + z^2) = 0$ find $\frac{\partial z}{\partial x}, \frac{\partial y}{\partial z}, \frac{\partial y}{\partial x}$

Solution:

Step 1: Let $F(x, y, z) = \sin(xyz) + x^3y^2z^2 + \ln(x^2 + y^2 + z^2)$.

Step 2: Compute partials:

$$F_x = yz\cos(xyz) + 3x^2y^2z^2 + \frac{2x}{x^2 + y^2 + z^2}$$
$$F_y = xz\cos(xyz) + 2x^3yz^2 + \frac{2y}{x^2 + y^2 + z^2}$$
$$F_z = xy\cos(xyz) + 2x^3y^2z + \frac{2z}{x^2 + y^2 + z^2}$$

Step 3: Apply formulas:

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \frac{\partial y}{\partial z} = -\frac{F_z}{F_y}, \frac{\partial y}{\partial x} = -\frac{F_x}{F_y}$$

$$\boxed{-\frac{F_x}{F_z}, -\frac{F_z}{F_y}, -\frac{F_x}{F_y}}$$