Assignment – 11

Topic: Partial Derivative

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Question 1: "The flow of the heat in a temperature field take place in the direction of maximum decrease of temperature." If T is a temperature field then find the direction of maximum change of temperature at given point.

(1) $T(x,y,z) = \frac{x}{x^2+y^2}$ at point (1, -1, 2)

(2) $T(x,y) = e^{x^2+y^2}\sin(2xy)$ at point $P(\frac{\pi}{2},0)$

Solution:

(1) Direction of maximum change = gradient ∇T.

Step 1: Compute partial derivatives:

$$T_x = \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$
$$T_y = \frac{0 - x(2y)}{(x^2+y^2)^2} = -\frac{2xy}{(x^2+y^2)^2}$$
$$T_z = 0$$

Step 2: At (1, -1, 2):

$$T_x = \frac{1-1}{(1+1)^2} = 0, T_y = -\frac{2(1)(-1)}{4} = \frac{2}{4} = \frac{1}{2}$$
$$\nabla T = (0,\frac{1}{2},0)$$

Direction of max decrease = opposite to gradient = $(0,-\frac{1}{2},0)$.

$$\boxed{(0,-\frac{1}{2},0)}$$

(2) $T(x,y) = e^{x^2+y^2}\sin(2xy)$

Step 1: Compute partials:

$$T_x = e^{x^2+y^2}[2x\sin(2xy) + 2y\cos(2xy)]$$
$$T_y = e^{x^2+y^2}[2y\sin(2xy) + 2x\cos(2xy)]$$

Step 2: At $(\pi/2,0)$:

$$T_x = e^{\pi^2/4}[\pi \sin\ 0 + 0] = 0$$
$$T_y = e^{\pi^2/4}[0 + \pi \cos\ 0] = \pi e^{\pi^2/4}$$
$$\nabla T = (0, \pi e^{\pi^2/4})$$

Direction of max decrease = $(0, -\pi e^{\pi^2/4})$.

$$\boxed{(0, -\pi e^{\pi^2/4})}$$

Question 2: If $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ and $|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$ then find grad$(|\vec{r}|^n)$.

Solution:

Step 1: Let $\phi = |\vec{r}|^n = (x^2 + y^2 + z^2)^{n/2}$

Step 2: Compute gradient:

$$\frac{\partial \phi}{\partial x} = \frac{n}{2}(x^2 + y^2 + z^2)^{n/2-1} \cdot 2x = nx\,|\vec{r}|^{n-2}$$

Similarly for y, z.

Step 3: So:

$$\text{grad}(|\vec{r}|^n) = n\,|\vec{r}|^{n-2}\,\vec{r}$$
$$\boxed{n\,|\vec{r}|^{n-2}\,\vec{r}}$$

Question 3: If $f(x, y, z) = e^{xyz} + \tan^{-1}(\frac{x}{y})$ then find grad(f) at point (1,1,1)

Solution:

Step 1: Compute partials:

$$f_x = yze^{xyz} + \frac{1}{1 + (x/y)^2} \cdot \frac{1}{y} = yze^{xyz} + \frac{y}{x^2 + y^2}$$
$$f_y = xze^{xyz} + \frac{1}{1 + (x/y)^2} \cdot (-\frac{x}{y^2}) = xze^{xyz} - \frac{x}{x^2 + y^2}$$

$$f_z = xye^{xyz}$$

Step 2: At (1,1,1):

$$f_x = 1 \cdot e^1 + \frac{1}{1+1} = e + \frac{1}{2}$$
$$f_y = 1 \cdot e^1 - \frac{1}{1+1} = e - \frac{1}{2}$$
$$f_z = 1 \cdot e^1 = e$$
$$\text{grad}(f) = (e + \frac{1}{2}, e - \frac{1}{2}, e)$$
$$\boxed{(e + \frac{1}{2}, e - \frac{1}{2}, e)}$$

Question 4: Explain directional derivative and find the directional derivative of $f(x, y, z) = 3e^x \cos(yz)$ at point $P(0,0,0)$ in the direction of $\vec{a} = 2\hat{\imath} + 2\hat{\jmath} - 2\hat{k}$.

Solution:

Step 1: Directional derivative = $\nabla f \cdot \hat{u}$, where $\hat{u}$ is unit vector in direction of $\vec{a}$.

Step 2: Compute ∇f:

$$f_x = 3e^x \cos(yz), f_y = -3ze^x \sin(yz), f_z = -3ye^x \sin(yz)$$

At (0,0,0):

$$\nabla f = (3 \cdot 1 \cdot 1,0,0) = (3,0,0)$$

Step 3: Unit vector $\hat{u}$:

$$|\vec{a}| = \sqrt{4+4+4} = \sqrt{12} = 2\sqrt{3}$$
$$\hat{u} = (\frac{2}{2\sqrt{3}}, \frac{2}{2\sqrt{3}}, -\frac{2}{2\sqrt{3}}) = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$$

Step 4: Dot product:

$$\nabla f \cdot \hat{u} = 3 \cdot \frac{1}{\sqrt{3}} + 0 + 0 = \sqrt{3}$$
$$\boxed{\sqrt{3}}$$

Question 5: Find the angle between the surface $x^2 + y^2 + z^2 = 9$ and $x^2 + y^2 - z = 3$ at point (2, -1, 2).

Solution:

Step 1: Let $f_1 = x^2 + y^2 + z^2 - 9$, $f_2 = x^2 + y^2 - z - 3$.
Gradients are normal to surfaces.

Step 2: Compute gradients:

$$\nabla f_1 = (2x, 2y, 2z) = (4, -2, 4) \text{ at } (2,-1,2)$$
$$\nabla f_2 = (2x, 2y, -1) = (4, -2, -1)$$

Step 3: Angle between surfaces = angle between normals:

$$\cos\theta = \frac{\nabla f_1 \cdot \nabla f_2}{|\nabla f_1||\nabla f_2|} = \frac{16 + 4 - 4}{\sqrt{16+4+16} \cdot \sqrt{16+4+1}} = \frac{16}{\sqrt{36} \cdot \sqrt{21}} = \frac{16}{6\sqrt{21}} = \frac{8}{3\sqrt{21}}$$

$$\theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$$

$$\boxed{\cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)}$$

Question 6: Find the equation of tangent plane and normal line to the surface $2xz^2 - 3xy - 4x = 7$ at point (1, -1, 2)

Solution:

Step 1: Let $F(x, y, z) = 2xz^2 - 3xy - 4x - 7$

Step 2: Compute gradient:

$$F_x = 2z^2 - 3y - 4, F_y = -3x, F_z = 4xz$$

At (1,-1,2):

$$F_x = 8 + 3 - 4 = 7, F_y = -3, F_z = 8$$
$$\nabla F = (7, -3, 8)$$

Step 3: Tangent plane:

$$7(x-1)-3(y+1)+8(z-2)=0$$
$$7x-7-3y-3+8z-16=0$$
$$7x-3y+8z=26$$

Step 4: Normal line:

$$\frac{x-1}{7}=\frac{y+1}{-3}=\frac{z-2}{8}$$

$$\boxed{7x-3y+8z=26,\frac{x-1}{7}=\frac{y+1}{-3}=\frac{z-2}{8}}$$

Question 7: For simple pendulum $T=2\pi\sqrt{\frac{l}{g}}$. Find the maximum error in T due to possible error 2.4% in l and 1% in g.

Solution:

Step 1: Relative error:

$$\frac{dT}{T}=\frac{1}{2}\frac{dl}{l}-\frac{1}{2}\frac{dg}{g}$$

Max relative error:

$$|\frac{dT}{T}|\le\frac{1}{2}|\frac{dl}{l}|+\frac{1}{2}|\frac{dg}{g}|=\frac{1}{2}(0.024)+\frac{1}{2}(0.01)=0.012+0.005=0.017$$

Max percentage error = 1.7%

$$\boxed{1.7\%}$$

Question 8: Find the local maximum and minimum values of $2(x^2-y^2)-x^4+y^4$.

Solution:

Step 1: Let $f(x,y)=2x^2-2y^2-x^4+y^4$

Step 2: Find critical points:

$$f_x = 4x - 4x^3 = 4x(1 - x^2) = 0 \Rightarrow x = 0, \pm 1$$
$$f_y = -4y + 4y^3 = 4y(y^2 - 1) = 0 \Rightarrow y = 0, \pm 1$$

Critical points: (0,0), (±1,0), (0,±1), (±1,±1)

Step 3: Second derivatives:

$$f_{xx} = 4 - 12x^2, f_{yy} = -4 + 12y^2, f_{xy} = 0$$

Discriminant $D = f_{xx}f_{yy} - (f_{xy})^2$

- At (0,0): $f_{xx} = 4, f_{yy} = -4, D = -16 < 0 \Rightarrow$ saddle
- At (±1,0): $f_{xx} = -8, f_{yy} = -4, D = 32 > 0, f_{xx} < 0 \Rightarrow$ local max, $f = 2 - 0 - 1 + 0 = 1$
- At (0,±1): $f_{xx} = 4, f_{yy} = 8, D = 32 > 0, f_{xx} > 0 \Rightarrow$ local min, $f = 0 - 2 - 0 + 1 = -1$
- At (±1,±1): $f_{xx} = -8, f_{yy} = 8, D = -64 < 0 \Rightarrow$ saddle

Step 4: Local max value = 1, Local min value = -1

$$\boxed{\text{Local max: 1, Local min: } -1}$$

Question 9: Find the shortest distance from origin to the surface $xyz^2 = 2$.

Solution:

Step 1: Minimize $d^2 = x^2 + y^2 + z^2$ subject to $xyz^2 = 2$.

Step 2: Use Lagrange multipliers: Let $F = x^2 + y^2 + z^2 + \lambda(xyz^2 - 2)$

Step 3: Partial derivatives:

$$F_x = 2x + \lambda yz^2 = 0(1)$$
$$F_y = 2y + \lambda xz^2 = 0(2)$$
$$F_z = 2z + \lambda 2xyz = 0(3)$$

Constraint: $xyz^2 = 2$

Step 4: From (1) and (2):
$2x + \lambda yz^2 = 0, 2y + \lambda xz^2 = 0$

Multiply first by x, second by y:
$2x^2 + \lambda xyz^2 = 0, 2y^2 + \lambda xyz^2 = 0$
Subtract: $2(x^2 - y^2) = 0 \Rightarrow x^2 = y^2 \Rightarrow x = \pm y$

Step 5: From (1) and (3):
$2x + \lambda yz^2 = 0, 2z + 2\lambda xyz = 0 \Rightarrow 1 + \lambda xy = 0 \Rightarrow \lambda = -\frac{1}{xy}$

Step 6: Substitute into (1): $2x - \frac{1}{xy} \cdot yz^2 = 0 \Rightarrow 2x - \frac{z^2}{x} = 0 \Rightarrow 2x^2 = z^2 \Rightarrow z^2 = 2x^2$

Step 7: Constraint: $x(\pm x)(2x^2) = 2 \Rightarrow \pm 2x^4 = 2 \Rightarrow x^4 = 1 \Rightarrow x = \pm 1$

Then $z^2 = 2 \Rightarrow z = \pm\sqrt{2}$

Step 8: Distance $d = \sqrt{1 + 1 + 2} = \sqrt{4} = 2$

$$\boxed{2}$$

Question 10: Prove that the rectangular solid of maximum volume that can be inscribed in a sphere is a cube.

Solution:

Step 1: Let sphere radius R, rectangular solid dimensions 2x, 2y, 2z.
Constraint: $x^2 + y^2 + z^2 = R^2$
Volume $V = 8xyz$

Step 2: Maximize $f = xyz$ subject to $x^2 + y^2 + z^2 = R^2$.
Lagrange: $\nabla f = \lambda \nabla g$:
$yz = \lambda 2x, xz = \lambda 2y, xy = \lambda 2z$

Step 3: Multiply first by x, second by y, third by z:
$xyz = \lambda 2x^2, xyz = \lambda 2y^2, xyz = \lambda 2z^2$
So $x^2 = y^2 = z^2 \Rightarrow x = y = z$

Thus solid is a cube.

$$\boxed{\text{Shown}}$$

Question 11: Find the numbers x, y, z such that $xyz = 8$ and $xy + yz + zx$ is maximum using Lagrange Multipliers method.

Solution:

Step 1: Maximize $f = xy + yz + zx$ subject to $g = xyz - 8 = 0$.

Step 2: Lagrange: $\nabla f = \lambda \nabla g$:

$$f_x = y + z = \lambda yz (1)$$
$$f_y = x + z = \lambda xz (2)$$
$$f_z = x + y = \lambda xy (3)$$

Constraint: $xyz = 8$

Step 3: From (1) and (2):
$y + z = \lambda yz, x + z = \lambda xz$
Multiply first by x, second by y:
$xy + xz = \lambda xyz, xy + yz = \lambda xyz$
So $xy + xz = xy + yz \Rightarrow xz = yz \Rightarrow x = y$ (assuming z≠0)

Similarly from (2) and (3): $x + z = \lambda xz, x + y = \lambda xy \Rightarrow x + z = x + y \Rightarrow z = y$

So $x = y = z$

Step 4: Constraint: $x^3 = 8 \Rightarrow x = 2$

So $x = y = z = 2$

$$\boxed{2,2,2}$$