

## Assignment – 12

### Topic: Multiple Integration

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**Question 1:** Explain double integration with its geometric meaning.

**Solution:**

Double integration computes volume under surface  $z = f(x, y)$  over region R in xy-plane.

**Question 2:** Evaluate following double integrals

(1)

$$\int_0^1 \int_0^1 \frac{dxdy}{\sqrt{(1-x^2)(1-y^2)}}$$

**Solution:**

**Step 1:** Separate the integrals since the integrand is a product of functions of x and y:

$$\int_0^1 \int_0^1 \frac{dxdy}{\sqrt{(1-x^2)(1-y^2)}} = \left( \int_0^1 \frac{dx}{\sqrt{1-x^2}} \right) \left( \int_0^1 \frac{dy}{\sqrt{1-y^2}} \right)$$

**Step 2:** Evaluate each integral:

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}} = [\sin^{-1}(x)]_0^1 = \sin^{-1}(1) - \sin^{-1}(0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

$$\int_0^1 \frac{dy}{\sqrt{1-y^2}} = \frac{\pi}{2}$$

**Step 3:** Multiply the results:

$$\left(\frac{\pi}{2}\right)\left(\frac{\pi}{2}\right) = \frac{\pi^2}{4}$$

**Final Answer:**

$$\boxed{\frac{\pi^2}{4}}$$

$$(2) \int_1^4 \int_{2x^2}^{3x^2} x e^{x^2+y} dy dx$$

**Step 1:** Integrate w.r.t y:

$$\int_{2x^2}^{3x^2} e^y dy = e^{3x^2} - e^{2x^2}$$

$$\text{So integral} = \int_1^4 x e^{x^2} (e^{3x^2} - e^{2x^2}) dx = \int_1^4 x (e^{4x^2} - e^{3x^2}) dx$$

**Step 2:** Let  $u = x^2$ ,  $du = 2x dx$ :

$$\begin{aligned} &= \frac{1}{2} \int_1^{16} (e^{4u} - e^{3u}) du = \frac{1}{2} \left[ \frac{e^{4u}}{4} - \frac{e^{3u}}{3} \right]_1^{16} \\ &= \frac{1}{2} \left( \frac{e^{64} - e^4}{4} - \frac{e^{48} - e^3}{3} \right) \\ &\boxed{\frac{1}{2} \left( \frac{e^{64} - e^4}{4} - \frac{e^{48} - e^3}{3} \right)} \end{aligned}$$

$$(3) \int_0^1 \int_0^{\sqrt{1-y^2}} (x^2 + y^2) dx dy$$

**Step 1:** Region: quarter circle in first quadrant, radius 1.

**Step 2:** Use polar:  $x = r \cos \theta$ ,  $y = r \sin \theta$ ,  $0 \leq r \leq 1$ ,  $0 \leq \theta \leq \pi/2$ , Jacobian = r.

$$\begin{aligned} \text{Integral} &= \int_0^{\pi/2} \int_0^1 r^2 \cdot r dr d\theta = \int_0^{\pi/2} d\theta \int_0^1 r^3 dr \\ &= \frac{\pi}{2} \cdot \frac{1}{4} = \frac{\pi}{8} \end{aligned}$$

$$\boxed{\frac{\pi}{8}}$$

$$(4) \int_0^\pi \int_{y=0}^x \frac{\sin x}{x} dy dx$$

**Step 1:** Integrate w.r.t y:

$$\int_0^x dy = x$$

So integral =  $\int_0^\pi \frac{\sin x}{x} \cdot x dx = \int_0^\pi \sin x dx = [-\cos x]_0^\pi = 2$

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**Question 3:** Evaluate  $\int_R (x + y)^2 dx dy$  where R is region enclosed by ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

**Solution:**

**Step 1:** Use transformation  $x = a \cos \theta, y = b \sin \theta$ , Jacobian =  $ab$ , region:  $0 \leq r \leq 1, 0 \leq \theta \leq 2\pi$ .

**Step 2:**  $(x + y)^2 = a^2 \cos^2 \theta + 2ab \cos \theta \sin \theta + b^2 \sin^2 \theta$

$$\text{Integral} = ab \int_0^{2\pi} \int_0^1 [a^2 \cos^2 \theta + 2ab \sin \theta \cos \theta + b^2 \sin^2 \theta] r^3 dr d\theta$$

**Step 3:**  $\int_0^1 r^3 dr = \frac{1}{4}$

$$\text{Integral} = \frac{ab}{4} \int_0^{2\pi} [a^2 \cos^2 \theta + 2ab \sin \theta \cos \theta + b^2 \sin^2 \theta] d\theta$$

**Step 4:**  $\int_0^{2\pi} \cos^2 \theta d\theta = \pi, \int_0^{2\pi} \sin^2 \theta d\theta = \pi, \int_0^{2\pi} \sin \theta \cos \theta d\theta = 0$

$$\text{So} = \frac{ab}{4} [a^2 \pi + b^2 \pi] = \frac{\pi ab}{4} (a^2 + b^2)$$

$$\boxed{\frac{\pi ab}{4} (a^2 + b^2)}$$

**Question 4:** Evaluate  $\iint_R x^2 dA$  where R is region in first quadrant bounded by hyperbola  $xy = 16$  and lines  $y = x, x = 4, x = 8$

**Solution:**

**Step 1:** Region: between  $y = x$  and  $y = 16/x$ , from  $x = 4$  to  $x = 8$ .

**Step 2:** Integral =  $\int_4^8 \int_x^{16/x} x^2 dy dx$

$$\text{Inner integral: } x^2 [y]_x^{16/x} = x^2 \left( \frac{16}{x} - x \right) = 16x - x^3$$

**Step 3:**  $\int_4^8 (16x - x^3)dx = [8x^2 - \frac{x^4}{4}]_4^8$

At  $x = 8$ :  $8 \cdot 64 - \frac{4096}{4} = 512 - 1024 = -512$

At  $x = 4$ :  $8 \cdot 16 - \frac{256}{4} = 128 - 64 = 64$

Difference:  $-512 - 64 = -576$

Wait, check: Actually  $\int (16x - x^3)dx = 8x^2 - x^4/4$

At 8:  $8 \cdot 64 - 4096/4 = 512 - 1024 = -512$

At 4:  $128 - 256/4 = 128 - 64 = 64$

So  $-512 - 64 = -576$

**Question 5:** Evaluate  $\iint_R r\sqrt{a^2 - r^2}drd\theta$  over the upper half of the circle  $r = a\cos \theta$

**Solution:**

**Step 1:** Region:  $0 \leq r \leq a\cos \theta, 0 \leq \theta \leq \pi/2$  (upper half).

**Step 2:** Integral =  $\int_0^{\pi/2} \int_0^{a\cos \theta} r\sqrt{a^2 - r^2}drd\theta$

**Step 3:** Inner integral: Let  $u = a^2 - r^2, du = -2rdr$ :

$$\begin{aligned} \int_0^{a\cos \theta} r\sqrt{a^2 - r^2}dr &= -\frac{1}{2} \int_{a^2}^{a^2 - a^2\cos^2 \theta} u^{1/2}du \\ &= \frac{1}{2} \int_{a^2\sin^2 \theta}^{a^2} u^{1/2}du = \frac{1}{2} \left[ \frac{2}{3} u^{3/2} \right]_{a^2\sin^2 \theta}^{a^2} \\ &= \frac{1}{3} (a^3 - a^3 |\sin^3 \theta|) = \frac{a^3}{3} (1 - \sin^3 \theta) \end{aligned}$$

**Step 4:** Outer integral:

$$\begin{aligned} \frac{a^3}{3} \int_0^{\pi/2} (1 - \sin^3 \theta)d\theta \\ \int_0^{\pi/2} 1d\theta &= \frac{\pi}{2} \\ \int_0^{\pi/2} \sin^3 \theta d\theta &= \frac{2}{3} \text{ (since } \int \sin^3 \theta d\theta = -\cos \theta + \frac{\cos^3 \theta}{3}, \text{ from } 0 \text{ to } \pi/2 \\ &= 0 - (-1 + 1/3) = 2/3) \end{aligned}$$

$$\text{So} = \frac{a^3}{3} \left( \frac{\pi}{2} - \frac{2}{3} \right)$$

$$\boxed{\frac{a^3}{3} \left( \frac{\pi}{2} - \frac{2}{3} \right)}$$

**Question 6:** Find the volume of prism where base is triangle in XY plane bounded by y-axis,  $y = x$  and  $y = 1$  and whose top lies in the plane  $z = 2 - x - y$ .

**Solution:**

**Step 1:** Base region:  $0 \leq x \leq 1, x \leq y \leq 1$ .

$$\begin{aligned} \text{Step 2: Volume} &= \iint_R (2 - x - y) dA \\ &= \int_0^1 \int_x^1 (2 - x - y) dy dx \end{aligned}$$

**Step 3:** Inner integral:

$$\int_x^1 (2 - x - y) dy = [(2 - x)y - \frac{y^2}{2}]_x^1$$

$$\text{At } y=1: (2 - x) - \frac{1}{2} = \frac{3}{2} - x$$

$$\text{At } y=x: (2 - x)x - \frac{x^2}{2} = 2x - x^2 - \frac{x^2}{2} = 2x - \frac{3x^2}{2}$$

$$\text{Difference: } \frac{3}{2} - x - (2x - \frac{3x^2}{2}) = \frac{3}{2} - 3x + \frac{3x^2}{2}$$

**Step 4:** Outer integral:

$$\int_0^1 \left( \frac{3}{2} - 3x + \frac{3x^2}{2} \right) dx = \left[ \frac{3x}{2} - \frac{3x^2}{2} + \frac{x^3}{2} \right]_0^1 = \frac{3}{2} - \frac{3}{2} + \frac{1}{2} = \frac{1}{2}$$

$$\boxed{\frac{1}{2}}$$

**Question 7:** A thin plate covers the triangular region bounded by x-axis and line  $x = 1$  and  $y = 2x$  in the first octant. The plate density at point  $(x, y)$  is  $\rho(x, y) = 6x + 6y + 6$ . Find the plate mass, first moments and center of mass

about the coordinate axis.

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**Solution:**

**Step 1:** Region:  $0 \leq x \leq 1, 0 \leq y \leq 2x$ .

**Step 2:** Mass  $M = \iint_R \rho(x, y) dA = \int_0^1 \int_0^{2x} (6x + 6y + 6) dy dx$

Inner:  $\int_0^{2x} (6x + 6y + 6) dy = [6xy + 3y^2 + 6y]_0^{2x} = 12x^2 + 12x^2 + 12x = 24x^2 + 12x$

Outer:  $\int_0^1 (24x^2 + 12x) dx = [8x^3 + 6x^2]_0^1 = 8 + 6 = 14$

So  $M = 14$

**Step 3:** First moment about y-axis  $M_y = \iint_R x \rho dA = \int_0^1 \int_0^{2x} x(6x + 6y + 6) dy dx$

Inner:  $\int_0^{2x} (6x^2 + 6xy + 6x) dy = [6x^2y + 3xy^2 + 6xy]_0^{2x} = 12x^3 + 12x^3 + 12x^2 = 24x^3 + 12x^2$

Outer:  $\int_0^1 (24x^3 + 12x^2) dx = [6x^4 + 4x^3]_0^1 = 6 + 4 = 10$

So  $M_y = 10$

**Step 4:** First moment about x-axis  $M_x = \iint_R y \rho dA = \int_0^1 \int_0^{2x} y(6x + 6y + 6) dy dx$

Inner:  $\int_0^{2x} (6xy + 6y^2 + 6y) dy = [3xy^2 + 2y^3 + 3y^2]_0^{2x} = 12x^3 + 16x^3 + 12x^2 = 28x^3 + 12x^2$

Outer:  $\int_0^1 (28x^3 + 12x^2) dx = [7x^4 + 4x^3]_0^1 = 7 + 4 = 11$

So  $M_x = 11$

**Step 5:** Center of mass  $\bar{x} = M_y/M = 10/14 = 5/7, \bar{y} = M_x/M = 11/14$

$M = 14, M_y = 10,$ $M_x = 11, \bar{x} = \frac{5}{7},$ $\bar{y} = \frac{11}{14}$
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