Assignment – 14

Topic: Multiple Integration

Question 1: Derive relation between Cartesian coordinate and spherical coordinate and cylindrical coordinate.

Solution:

Cylindrical coordinates:

$$x = r\cos\ \theta, y = r\sin\ \theta, z = z$$

$$\boxed{x = \mathrm{r}\cos\theta\,,\, y = \mathrm{r}\sin\theta\,,\, z = z}$$

Spherical coordinates:

$$x = \rho\sin\ \phi\cos\ \theta, y = \rho\sin\ \phi\sin\ \theta, z = \rho\cos\ \phi$$

$$\boxed{x = \rho\sin\phi\cos\theta\,,\, y = \rho\sin\phi\sin\theta\,,\, z = \rho\cos\ \phi}$$

Question 2: Evaluate following triple integrals

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(1) $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dzdydx}{\sqrt{1-x^2-y^2-z^2}}$

Step 1: Region: unit octant (first octant of sphere radius 1).

Step 2: Use spherical: $x = \rho\sin\ \phi\cos\ \theta, y = \rho\sin\ \phi\sin\ \theta, z = \rho\cos\ \phi, 0 \leq \rho \leq 1, 0 \leq \phi \leq \pi/2, 0 \leq \theta \leq \pi/2$, Jacobian = $\rho^2\sin\ \phi$.

Step 3: Integrand: $\frac{1}{\sqrt{1-\rho^2}}$

Integral = $\int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 \frac{1}{\sqrt{1-\rho^2}} \cdot \rho^2\sin\ \phi d\rho d\phi d\theta$

Step 4: Separate:

$$\left(\int_0^{\pi/2} d\theta\right)\left(\int_0^{\pi/2} \sin\ \phi d\phi\right)\left(\int_0^1 \frac{\rho^2}{\sqrt{1-\rho^2}} d\rho\right)$$

$$= \frac{\pi}{2} \cdot [-\cos\ \phi]_0^{\pi/2} \cdot \int_0^1 \frac{\rho^2}{\sqrt{1-\rho^2}} d\rho$$

$$= \frac{\pi}{2} \cdot (0 - (-1)) \cdot \int_0^1 \frac{\rho^2}{\sqrt{1-\rho^2}} d\rho = \frac{\pi}{2} \cdot \int_0^1 \frac{\rho^2}{\sqrt{1-\rho^2}} d\rho$$

Step 5: Let $\rho = \sin\ t, d\rho = \cos\ tdt, 0 \le t \le \pi/2$:

$$\int \frac{\sin^2 t}{\cos t} \cdot \cos\ tdt = \int \sin^2 tdt = \int \frac{1-\cos\ 2t}{2} dt = \frac{t}{2} - \frac{\sin\ 2t}{4}$$

Evaluate 0 to π/2: $\frac{\pi}{4} - 0 = \frac{\pi}{4}$

So integral = $\frac{\pi}{2} \cdot \frac{\pi}{4} = \frac{\pi^2}{8}$

$$\boxed{\frac{\pi^2}{8}}$$

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(2)

$$\int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{a\sin\ \theta} \int_{z=0}^{\frac{a^2-r^2}{a}} rdzdrd\theta$$

Solution:

Step 1: Evaluate the innermost integral with respect to z:

$$\int_0^{\frac{a^2-r^2}{a}} dz = \frac{a^2-r^2}{a}$$

Step 2: Substitute back:

$$\int_0^{\frac{\pi}{2}} \int_0^{asi\quad \theta} r \cdot \frac{a^2-r^2}{a} drd\theta = \frac{1}{a} \int_0^{\frac{\pi}{2}} \int_0^{a\sin\ \theta} (a^2 r - r^3) drd\theta$$

Step 3: Evaluate the inner integral with respect to r:

$$\int_0^{a\sin\ \theta} (a^2 r - r^3) dr = [\frac{a^2 r^2}{2} - \frac{r^4}{4}]_0^{as\quad \theta}$$

$$= \frac{a^4 \sin^2 \theta}{2} - \frac{a^4 \sin^4 \theta}{4} = \frac{a^4}{4}(2\sin^2 \theta - \sin^4 \theta)$$

Step 4: Substitute back:

$$\frac{1}{a}\int_0^{\frac{\pi}{2}} \frac{a^4}{4}(2\sin^2 \theta - \sin^4 \theta)d\theta = \frac{a^3}{4}\int_0^{\frac{\pi}{2}} (2\sin^2 \theta - \sin^4 \theta)d\theta$$

Step 5: Evaluate the trigonometric integrals:

$$\int_0^{\frac{\pi}{2}} \sin^2 \theta d\theta = \frac{\pi}{4}, \int_0^{\frac{\pi}{2}} \sin^4 \theta d\theta = \frac{3\pi}{16}$$

$$\int_0^{\frac{\pi}{2}} (2\sin^2 \theta - \sin^4 \theta)d\theta = 2 \cdot \frac{\pi}{4} - \frac{3\pi}{16} = \frac{\pi}{2} - \frac{3\pi}{16} = \frac{8\pi - 3\pi}{16} = \frac{5\pi}{16}$$

Step 6: Multiply:

$$\frac{a^3}{4} \cdot \frac{5\pi}{16} = \frac{5\pi a^3}{64}$$

Final Answer:

$$\boxed{\frac{5\pi a^3}{64}}$$

Question 3: Evaluate $\iiint_D \frac{dV}{(x^2+y^2+z^2)^{3/2}}$ where D is the region bounded by the spheres $x^2 + y^2 + z^2 = a^2$ and $x^2 + y^2 + z^2 = b^2$ with $0 < a < b$.

Solution:

Step 1: Use spherical: ρ from a to b, $0 \le \phi \le \pi$, $0 \le \theta \le 2\pi$, Jacobian $= \rho^2 \sin \phi$.

Step 2: Integrand: $\frac{1}{\rho^3}$

Integral = $\int_0^{2\pi} \int_0^{\pi} \int_a^b \frac{1}{\rho^3} \cdot \rho^2 \sin \phi d\rho d\phi d\theta$

$$= \int_0^{2\pi} d\theta \int_0^{\pi} \sin \phi d\phi \int_a^b \frac{1}{\rho} d\rho$$

Step 3: $\int_0^{2\pi} \quad d\theta = 2\pi, \int_0^{\pi} \quad \sin\ \phi d\phi = 2, \int_a^b \quad \frac{1}{\rho} d\rho = \ln{(b/a)}$

So = $2\pi \cdot 2 \cdot \ln{(b/a)} = 4\pi \ln{(b/a)}$

$$\boxed{4\pi \ln{(\frac{b}{a})}}$$

Question 4: Find the volume of the region B bounded by paraboloid $z = 4 - x^2 - y^2$ and XY plane.

Solution:

Step 1: In XY plane, $z = 0 \Rightarrow 4 - x^2 - y^2 = 0 \Rightarrow x^2 + y^2 = 4$ (circle radius 2).

Step 2: Volume = $\iint_{x^2+y^2 \leq 4} \quad (4 - x^2 - y^2) dA$

Step 3: Use polar: $0 \leq r \leq 2, 0 \leq \theta \leq 2\pi$:

Volume = $\int_0^{2\pi} \quad \int_0^2 \quad (4 - r^2) \cdot r dr d\theta$

Step 4: Inner: $\int_0^2 \quad (4r - r^3) dr = [2r^2 - \frac{r^4}{4}]_0^2 = 8 - 4 = 4$

Outer: $\int_0^{2\pi} \quad 4 d\theta = 8\pi$

$$\boxed{8\pi}$$

Question 5: Evaluate $\iiint_B \quad \sqrt{x^2 + y^2 + z^2} dV$ where B is the region bounded by the plane $z = 3$ and the cone $z = \sqrt{x^2 + y^2}$.

Solution:

Step 1: Region: between cone and plane z=3. In cylindrical: $0 \leq z \leq 3, 0 \leq r \leq z, 0 \leq \theta \leq 2\pi$.

Step 2: Integrand $\sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2}$, Jacobian = r.

Integral = $\int_0^{2\pi} \quad \int_0^3 \quad \int_0^z \quad \sqrt{r^2 + z^2} \cdot r dr dz d\theta$

Step 3: Inner: Let $u = r^2 + z^2$, $du = 2rdr$, limits: $r = 0 \Rightarrow u = z^2$, $r = z \Rightarrow u = 2z^2$:

$$\int_0^z r\sqrt{r^2 + z^2}dr = \frac{1}{2}\int_{z^2}^{2z^2} u^{1/2}du = \frac{1}{2} \cdot \frac{2}{3}[u^{3/2}]_{z^2}^{2z^2} = \frac{1}{3}((2z^2)^{3/2} - (z^2)^{3/2})$$

$$= \frac{1}{3}(2\sqrt{2}z^3 - z^3) = \frac{z^3}{3}(2\sqrt{2} - 1)$$

Step 4: Middle: $\int_0^3 \frac{z^3}{3}(2\sqrt{2} - 1)dz = \frac{2\sqrt{2}-1}{3} \cdot \frac{z^4}{4} |_0^3 = \frac{2\sqrt{2}-1}{3} \cdot \frac{81}{4} = \frac{27(2\sqrt{2}-1)}{4}$

Step 5: Outer: $\int_0^{2\pi} d\theta = 2\pi$

So = $2\pi \cdot \frac{27(2\sqrt{2}-1)}{4} = \frac{27\pi}{2}(2\sqrt{2} - 1)$

$$\boxed{\frac{27\pi}{2}(2\sqrt{2} - 1)}$$

Question 6: Find the volume of "ice cream cone" cut from the solid sphere $\rho \leq 1$ by the cone $\phi = \frac{\pi}{3}$.

Solution:

Step 1: Spherical coordinates: $0 \leq \rho \leq 1$, $0 \leq \phi \leq \pi/3$, $0 \leq \theta \leq 2\pi$, Jacobian = $\rho^2 \sin\ \phi$.

Step 2: Volume = $\int_0^{2\pi} \int_0^{\pi/3} \int_0^1 \rho^2 \sin\ \phi d\rho d\phi d\theta$

Step 3: Inner: $\int_0^1 \rho^2 d\rho = \frac{1}{3}$

Middle: $\int_0^{\pi/3} \sin\ \phi d\phi = [-\cos\ \phi]_0^{\pi/3} = -\frac{1}{2} + 1 = \frac{1}{2}$

Outer: $\int_0^{2\pi} d\theta = 2\pi$

So = $2\pi \cdot \frac{1}{2} \cdot \frac{1}{3} = \frac{\pi}{3}$

$$\boxed{\frac{\pi}{3}}$$

Question 7: Show that the volume of sphere of radius r is $\frac{4}{3}\pi r^3$ by using (i) single integral (ii) double integral (iii) triple integral

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Solution:

(i) Single integral (disk method):
Volume = $\int_{-r}^{r} \pi[\sqrt{r^2 - x^2}]^2 dx = \pi \int_{-r}^{r} (r^2 - x^2) dx$

$$= \pi[r^2 x - \frac{x^3}{3}]_{-r}^{r} = \pi(2r^3 - \frac{2r^3}{3}) = \frac{4\pi r^3}{3}$$

(ii) Double integral (surface area):
Surface area $A = 4\pi r^2$, but for volume via double integral in cylindrical:
Volume = $\int_0^{2\pi} \int_0^r \sqrt{r^2 - \rho^2} \cdot \rho d\rho d\theta$
Let $u = r^2 - \rho^2, du = -2\rho d\rho$:
Inner = $-\frac{1}{2}\int_{r^2}^{0} u^{1/2} du = \frac{1}{2} \cdot \frac{2}{3}[u^{3/2}]_0^{r^2} = \frac{1}{3}r^3$
Outer = $\int_0^{2\pi} d\theta = 2\pi$
So = $2\pi \cdot \frac{1}{3}r^3 = \frac{2\pi r^3}{3}$ — this is half? Actually this gives volume of hemisphere. So full volume = $2 \cdot \frac{2\pi r^3}{3} = \frac{4\pi r^3}{3}$.

(iii) Triple integral (spherical):
Volume = $\int_0^{2\pi} \int_0^{\pi} \int_0^r \rho^2 \sin\, \phi d\rho d\phi d\theta$

$$= (\int_0^{2\pi} d\theta)(\int_0^{\pi} \sin\, \phi d\phi)(\int_0^r \rho^2 d\rho)$$

$$= 2\pi \cdot 2 \cdot \frac{r^3}{3} = \frac{4\pi r^3}{3}$$

$$\boxed{\frac{4\pi r^3}{3}}$$

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