Subject Name & Code:

MATHEMATICS II- BE02R00011

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Assignment – 2

1. Solve Linear System Using Gauss-Jordan Method

(i) Given:

$$\begin{cases} x + 2y - z = -1 & (1) \\ x + 8y + 2z = 28 & (2) \\ 4x + 9y - z = 14 & (3) \end{cases}$$

Augmented matrix:

$$\begin{bmatrix} 1 & 2 & -1 & -1 \\ 1 & 8 & 2 & 28 \\ 4 & 9 & -1 & 14 \end{bmatrix}$$

Step 1: $R_2 \leftarrow R_2 - R_1$

$$\begin{bmatrix} 1 & 2 & -1 & -1 \\ 0 & 6 & 3 & 29 \\ 4 & 9 & -1 & 14 \end{bmatrix}$$

Step 2: $R_3 \leftarrow R_3 - 4R_1$

$$\begin{bmatrix} 1 & 2 & -1 & -1 \\ 0 & 6 & 3 & 29 \\ 0 & 1 & 3 & 18 \end{bmatrix}$$

Step 3: Swap R_2 and R_3 for easier pivot:

$$\begin{bmatrix} 1 & 2 & -1 & -1 \\ 0 & 1 & 3 & 18 \\ 0 & 6 & 3 & 29 \end{bmatrix}$$

Step 4: $R_3 \leftarrow R_3 - 6R_2$

$$\begin{bmatrix} 1 & 2 & -1 & -1 \\ 0 & 1 & 3 & 18 \\ 0 & 0 & -15 & -79 \end{bmatrix}$$

Step 5: $R_3 \leftarrow -\frac{1}{15}R_3$

$$\begin{bmatrix} 1 & 2 & -1 & -1 \\ 0 & 1 & 3 & 18 \\ 0 & 0 & 1 & \frac{79}{15} \end{bmatrix}$$

Step 6: Back elimination (Gauss-Jordan):

- $R_2 \leftarrow R_2 - 3R_3$
- $R_1 \leftarrow R_1 + R_3$

After calculations:

$$\begin{bmatrix} 1 & 2 & 0 & \frac{-1 + 79/15}{?} \\ 0 & 1 & 0 & 18 - 3 \cdot \frac{79}{15} \\ 0 & 0 & 1 & \frac{79}{15} \end{bmatrix}$$

Let's compute carefully:

$$18 - 3 \cdot \frac{79}{15} = 18 - \frac{237}{15} = \frac{270}{15} - \frac{237}{15} = \frac{33}{15} = \frac{11}{5}$$

$R_1: -1 + \frac{79}{15} = \frac{-15}{15} + \frac{79}{15} = \frac{64}{15}$ (with z removed, but we need to eliminate y too first).

Better to proceed systematically:

From Step 5 matrix:
$R_2 \leftarrow R_2 - 3R_3$:

$$y + 3z = 18 \Rightarrow y = 18 - 3 \cdot \frac{79}{15} = \frac{11}{5}$$

$R_1 \leftarrow R_1 + R_3$:

$$x + 2y - z + z = x + 2y = -1 + \frac{79}{15} = \frac{64}{15}$$

Then $x = \frac{64}{15} - 2 \cdot \frac{11}{5} = \frac{64}{15} - \frac{22}{5} = \frac{64}{15} - \frac{66}{15} = -\frac{2}{15}$

$$z = \frac{79}{15}$$

Final Solution:

$$\boxed{x = -\frac{2}{15}, y = \frac{11}{5}, z = \frac{79}{15}}$$

1. (ii) Solve the following linear system using Gauss-Jordan method

Given:

$$\begin{cases} x + y + z = 6 & (1) \\ x + 2y + 3z = 14 & (2) \\ 2x + 4y + 7z = 30 & (3) \end{cases}$$

Step 1: Write augmented matrix

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 14 \\ 2 & 4 & 7 & 30 \end{bmatrix}$$

Step 2: Apply row operations

- $R_2 \leftarrow R_2 - R_1$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 2 & 4 & 7 & 30 \end{bmatrix}$$

- $R_3 \leftarrow R_3 - 2R_1$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 2 & 5 & 18 \end{bmatrix}$$

- $R_3 \leftarrow R_3 - 2R_2$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Now matrix is in **row-echelon form**. Continue to **reduced row-echelon form** (Gauss-Jordan):

Step 3: $R_2 \leftarrow R_2 - 2R_3$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Step 4: $R_1 \leftarrow R_1 - R_3$

$$\begin{bmatrix} 1 & 1 & 0 & 4 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Step 5: $R_1 \leftarrow R_1 - R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Solution:

$$x = 0, y = 4, z = 2$$

Final Answer:

$$\boxed{(x, y, z) = (0,4,2)}$$

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2. Investigate Values of a and b for Solution Types

Given:

$$\begin{cases} x + y + z = 6 & (1) \\ x + 2y + 3z = 10 & (2) \\ x + 2y + az = b & (3) \end{cases}$$

Augmented matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & a & b \end{bmatrix}$$

Row operations:

- $R_2 \leftarrow R_2 - R_1$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 1 & 2 & a & b \end{bmatrix}$$

- $R_3 \leftarrow R_3 - R_1$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & a-1 & b-6 \end{bmatrix}$$

- $R_3 \leftarrow R_3 - R_2$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & a-3 & b-10 \end{bmatrix}$$

Analysis:
Let $d = a - 3$, $k = b - 10$.

1. **No solution:** if $d = 0$ and $k \neq 0 \rightarrow a = 3, b \neq 10$.
2. **Infinite solutions:** if $d = 0$ and $k = 0 \rightarrow a = 3, b = 10$.
3. **Unique solution:** if $d \neq 0 \rightarrow a \neq 3$ (any b).

Summary:

$$\boxed{\begin{array}{l} \text{No solution: } a = 3,\ b \neq 10 \\ \text{Infinite solutions: } a = 3,\ b = 10 \\ \text{Unique solution: } a \neq 3 \end{array}}$$

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3. Gauss Elimination vs. Gauss-Jordan

Given:

$$\begin{cases} x + y + z = 6 & (1) \\ x + 2y + 3z = 14 & (2) \\ 2x + 4y + 7z = 30 & (3) \end{cases}$$

Augmented matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 14 \\ 2 & 4 & 7 & 30 \end{bmatrix}$$

Gauss Elimination (Row Echelon Form):

- $R_2 \leftarrow R_2 - R_1$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 2 & 4 & 7 & 30 \end{bmatrix}$$

- $R_3 \leftarrow R_3 - 2R_1$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 2 & 5 & 18 \end{bmatrix}$$

- $R_3 \leftarrow R_3 - 2R_2$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Back substitution:

$$z = 2$$
$$y + 2z = 8 \Rightarrow y = 4$$
$$x + y + z = 6 \Rightarrow x = 0$$

Solution: (0، 4، 2)

Gauss-Jordan (Reduced Row Echelon Form):

From last REF:

- $R_2 \leftarrow R_2 - 2R_3$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

- $R_1 \leftarrow R_1 - R_2 - R_3$ directly:

$$x + y + z - y - z = x$$
$$6 - 4 - 2 = 0$$

Thus:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Solution: (0، 4، 2) — same as Gauss elimination.

Comparison:

- **Gauss elimination** stops at upper triangular form and uses back substitution.
- **Gauss-Jordan** continues to diagonal form, giving solution directly without substitution.
- Gauss-Jordan involves more operations but is more straightforward for reading solutions.

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4. Eigenvalues of A, A^{-1}, A^T, A^2

Given:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}$$

Eigenvalues of A:
Solve $\det(A - \lambda I) = 0$:

$$\begin{vmatrix} 1-\lambda & 2 \\ 0 & 4-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda) = 0$$

$$\lambda_1 = 1, \ \lambda_2 = 4$$

Eigenvalues of A^{-1}:
Eigenvalues are $\frac{1}{\lambda_i}$:

$$1, \ \frac{1}{4}$$

Eigenvalues of A^T:
Same as A: 1,4.

Eigenvalues of A^2:
Eigenvalues are λ_i^2:

$$1,\ 16$$

Summary:

$$\boxed{\begin{array}{r}\lambda(A) = \{1,4\} \\ \lambda(A^{-1}) = \{1, \frac{1}{4}\} \\ \lambda(A^T) = \{1,4\} \\ \lambda(A^2) = \{1,16\}\end{array}}$$

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5. Eigenvalues and Eigenvectors of Given Matrix

(i) Given:

$$A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

Characteristic equation:

$$\det(A - \lambda I) = \left| \begin{matrix} 4-\lambda & 0 & 1 \\ -2 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{matrix} \right|$$

Expand along first row:

$$(4-\lambda)[(1-\lambda)^2 - 0] - 0 + 1[0 - (-2)(1-\lambda)]$$

$= (4-\lambda)(1-\lambda)^2 + 2(1-\lambda)$

Factor $(1-\lambda)$:

$$(1-\lambda)[(4-\lambda)(1-\lambda) + 2]$$

$= (1-\lambda)[(4-\lambda)(1-\lambda) + 2]$

Compute inside: $(4-\lambda)(1-\lambda) = 4 - 4\lambda - \lambda + \lambda^2 = \lambda^2 - 5\lambda + 4$
Add 2: $\lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3)$

Thus:

$$\det(A - \lambda I) = (1-\lambda)(\lambda-2)(\lambda-3) = 0$$

Eigenvalues: $\lambda = 1,2,3$.

Eigenvectors:

- For $\lambda = 1$: Solve $(A - I)\mathbf{v} = 0$

$$\begin{bmatrix} 3 & 0 & 1 \\ -2 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix} \mathbf{v} = 0$$

From second row: $-2v_1 = 0 \Rightarrow v_1 = 0$
First row: $3v_1 + v_3 = 0 \Rightarrow v_3 = 0$
v_2 free: $\mathbf{v} = t(0,1,0)$.

- For $\lambda = 2$: Solve $(A - 2I)\mathbf{v} = 0$

$$\begin{bmatrix} 2 & 0 & 1 \\ -2 & -1 & 0 \\ -2 & 0 & -1 \end{bmatrix} \mathbf{v} = 0$$

Row 1: $2v_1 + v_3 = 0 \Rightarrow v_3 = -2v_1$
Row 2: $-2v_1 - v_2 = 0 \Rightarrow v_2 = -2v_1$
Let $v_1 = 1$: $\mathbf{v} = (1, -2, -2)$.

- For $\lambda = 3$: Solve $(A - 3I)\mathbf{v} = 0$

$$\begin{bmatrix} 1 & 0 & 1 \\ -2 & -2 & 0 \\ -2 & 0 & -2 \end{bmatrix} \mathbf{v} = 0$$

Row 1: $v_1 + v_3 = 0 \Rightarrow v_3 = -v_1$
Row 2: $-2v_1 - 2v_2 = 0 \Rightarrow v_2 = -v_1$
Let $v_1 = 1$: $\mathbf{v} = (1, -1, -1)$.

Eigenvectors (normalized form):

$$\boxed{\begin{aligned} \lambda_1 = 1&: \mathbf{v}_1 = (0,1,0) \\ \lambda_2 = 2&: \mathbf{v}_2 = (1,-2,-2) \\ \lambda_3 = 3&: \mathbf{v}_3 = (1,-1,-1) \end{aligned}}$$

5. (ii) Eigenvalues and Eigenvectors of

Given:

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$$

Step 1: Find eigenvalues

Solve $\det(A - \lambda I) = 0$:

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 2 & 2 \\ 0 & 2-\lambda & 1 \\ -1 & 2 & 2-\lambda \end{bmatrix}$$

Determinant expansion (using first row):

$$\begin{aligned} \det(A - \lambda I) &= (1-\lambda) \begin{vmatrix} 2-\lambda & 1 \\ 2 & 2-\lambda \end{vmatrix} - 2 \begin{vmatrix} 0 & 1 \\ -1 & 2-\lambda \end{vmatrix} + 2 \begin{vmatrix} 0 & 2-\lambda \\ -1 & 2 \end{vmatrix} \\ &= (1-\lambda)[(2-\lambda)(2-\lambda) - 2 \cdot 1] - 2[0 \cdot (2-\lambda) - (-1) \cdot 1] + 2[0 \cdot 2 - (-1)(2-\lambda)] \end{aligned}$$

Compute each term:

1. $(2-\lambda)^2 - 2 = \lambda^2 - 4\lambda + 4 - 2 = \lambda^2 - 4\lambda + 2$
   Multiply by $(1-\lambda)$:

$$(1-\lambda)(\lambda^2 - 4\lambda + 2) = -\lambda^3 + 5\lambda^2 - 6\lambda + 2$$

2. $-2 \times [0 - (-1)] = -2 \times 1 = -2$
3. $2 \times [0 + (2-\lambda)] = 2(2-\lambda) = 4 - 2\lambda$

Sum them:

$$\begin{aligned} &(-\lambda^3 + 5\lambda^2 - 6\lambda + 2) + (-2) + (4 - 2\lambda) \\ &= -\lambda^3 + 5\lambda^2 - 8\lambda + 4 \end{aligned}$$

Set equal to zero:

$$-\lambda^3 + 5\lambda^2 - 8\lambda + 4 = 0$$
$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

Test $\lambda = 1$: $1 - 5 + 8 - 4 = 0$, so $\lambda = 1$ is a root.

Divide by $(\lambda - 1)$:

$$(\lambda - 1)(\lambda^2 - 4\lambda + 4) = 0$$
$$(\lambda - 1)(\lambda - 2)^2 = 0$$

Eigenvalues:

$$\boxed{\lambda_1 = 1, \lambda_2 = 2 \text{ (double root)}}$$

Step 2: Eigenvectors for $\lambda = 1$

Solve $(A - I)\mathbf{v} = 0$:

$$A - I = \begin{bmatrix} 0 & 2 & 2 \\ 0 & 1 & 1 \\ -1 & 2 & 1 \end{bmatrix}$$

Row reduce:

- Swap R_1 and R_3 for convenience:

$$\begin{bmatrix} -1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$$

- $R_3 \leftarrow R_3 - 2R_2$:

$$\begin{bmatrix} -1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

- $R_1 \leftarrow -R_1$:

$$\begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

- $R_1 \leftarrow R_1 + 2R_2$:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Equations:

$$v_1 + v_3 = 0 \Rightarrow v_1 = -v_3$$
$$v_2 + v_3 = 0 \Rightarrow v_2 = -v_3$$

Let $v_3 = t$, then $v_1 = -t$, $v_2 = -t$:

Eigenvector:

$$\mathbf{v}_1 = t\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, t \neq 0$$

Normalized choice: $\mathbf{v}_1 = (-1, -1, 1)$.

Step 3: Eigenvectors for $\lambda = 2$

Solve $(A - 2I)\mathbf{v} = 0$:

$$A - 2I = \begin{bmatrix} -1 & 2 & 2 \\ 0 & 0 & 1 \\ -1 & 2 & 0 \end{bmatrix}$$

Row reduce:

- Already last row $(-1,2,0)$, first row $(-1,2,2)$.

$R_3 \leftarrow R_3 - R_1$:

$$\begin{bmatrix} -1 & 2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

$R_3 \leftarrow R_3 + 2R_2$:

$$\begin{bmatrix} -1 & 2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

From R_2: $v_3 = 0$
From R_1: $-v_1 + 2v_2 = 0 \Rightarrow v_1 = 2v_2$
v_2 free.

Let $v_2 = s$, then $v_1 = 2s$, $v_3 = 0$:

One eigenvector: $\mathbf{v}_2 = (2,1,0)$.

Since $\lambda = 2$ has multiplicity 2,
rank $(A - 2I) = 2 \Rightarrow$ nullity $= 3 - 2 = 1$, so only **one** eigenvector.
Thus A is **defective** at $\lambda = 2$.

Final Answer:

$$\boxed{\begin{array}{l} \text{Eigenvalues: } \lambda = 1,\ 2 \text{ (double)} \\ \lambda = 1: \quad \mathbf{v} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \\ \lambda = 2: \quad \mathbf{v} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \quad \text{(only one independent eigenvector)} \end{array}}$$

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6. Find det (A) from Characteristic Polynomial

Given: Characteristic polynomial

$$P(\lambda) = \lambda^3 - 2\lambda^2 + \lambda + 5$$

For an $n \times n$ matrix A,

$$P(\lambda) = \det(A - \lambda I) = (-1)^n \lambda^n + \cdots + \det(A)$$

For $n = 3$:

$$P(\lambda) = -\lambda^3 + (\text{trace})\lambda^2 - (\text{sum of principal minors})\lambda + \det(A)$$

But matching with given:

$$\lambda^3 - 2\lambda^2 + \lambda + 5 = (-1)^3\lambda^3 + \cdots + \det(A)$$

So $(-1)^3 = -1$ times λ^3 coefficient in $P(\lambda)$? Wait: Actually $P(\lambda) = \det(A - \lambda I)$ expands as:

$$P(\lambda) = (-1)^n\lambda^n + (-1)^{n-1}(\text{trace})\lambda^{n-1} + \cdots + \det(A)$$

For $n = 3$:

$$P(\lambda) = -\lambda^3 + (\text{trace})\lambda^2 - (\text{sum of principal minors})\lambda + \det(A)$$

Given $P(\lambda) = \lambda^3 - 2\lambda^2 + \lambda + 5$, multiply by -1 to match leading term:
Multiply $P(\lambda)$ by -1: $-\lambda^3 + 2\lambda^2 - \lambda - 5$
Compare with $-\lambda^3 + a\lambda^2 + b\lambda + \det(A)$:
$\det(A) = -5$.

Thus:

$$\boxed{\det(A) = -5}$$