Subject Name & Code:

MATHEMATICS II- BE02R00011

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Assignment – 3

1. State Cayley-Hamilton Theorem and Applications

Cayley-Hamilton Theorem:
Every square matrix satisfies its own characteristic equation. That is, if A is an $n \times n$ matrix and $p(\lambda) = \det(A - \lambda I)$ is its characteristic polynomial, then $p(A) = 0$.

(a) Verify for $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

Given:

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

Step 1: Characteristic polynomial

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) - 8$$
$$= \lambda^2 - 4\lambda + 3 - 8 = \lambda^2 - 4\lambda - 5$$

So $p(\lambda) = \lambda^2 - 4\lambda - 5$.

Step 2: Verify $p(A) = 0$

$$A^2 = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1+8 & 4+12 \\ 2+6 & 8+9 \end{bmatrix} = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix}$$
$$p(A) = A^2 - 4A - 5I$$
$$= \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} - 4\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} - \begin{bmatrix} 4 & 16 \\ 8 & 12 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Verified.

(b) Find A^{-1} using Cayley-Hamilton

From $A^2 - 4A - 5I = 0$, multiply by A^{-1}:

$$A - 4I - 5A^{-1} = 0$$
$$5A^{-1} = A - 4I$$
$$A^{-1} = \frac{1}{5}\left(\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}\right) = \frac{1}{5}\begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix}$$

(c) Find A^3

From $A^2 = 4A + 5I$,

$$A^3 = A \cdot A^2 = A(4A + 5I) = 4A^2 + 5A$$
$$= 4(4A + 5I) + 5A = 16A + 20I + 5A = 21A + 20I$$
$$= 21\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} + 20\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 41 & 84 \\ 42 & 83 \end{bmatrix}$$

(d) Express $A^4 - 7A^3 + 11A^2 - A - 10I$ as linear polynomial in A

From Cayley-Hamilton, $A^2 = 4A + 5I$.

Compute powers:

$$A^3 = 21A + 20I$$
$$A^4 = A \cdot A^3 = A(21A + 20I) = 21A^2 + 20A = 21(4A + 5I) + 20A = 84A + 105I + 20A$$
$$= 104A + 105I$$

Now substitute into expression:

$$A^4 - 7A^3 + 11A^2 - A - 10I$$
$$= [104A + 105I] - 7[21A + 20I] + 11[4A + 5I] - A - 10I$$
$$= 104A + 105I - 147A - 140I + 44A + 55I - A - 10I$$
$$= (104 - 147 + 44 - 1)A + (105 - 140 + 55 - 10)I$$
$$= 0 \cdot A + 10I = 10I$$

Thus, the expression simplifies to $10I$, a constant (degree 0 polynomial in A).

Final for Q1:

- Verified Cayley-Hamilton.
- $A^{-1} = \frac{1}{5}\begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix}$
- $A^3 = \begin{bmatrix} 41 & 84 \\ 42 & 83 \end{bmatrix}$
- $A^4 - 7A^3 + 11A^2 - A - 10I = 10I$

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2. Find A^{-1} using Cayley-Hamilton theorem

(i) $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

Step 1: Find characteristic polynomial

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -1 \\ 2 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) + 2$$
$$= \lambda^2 - 4\lambda + 3 + 2 = \lambda^2 - 4\lambda + 5$$

So $p(\lambda) = \lambda^2 - 4\lambda + 5$.

Step 2: Apply Cayley-Hamilton theorem

$$A^2 - 4A + 5I = 0$$

Step 3: Solve for A^{-1}
Multiply by A^{-1}:

$$A - 4I + 5A^{-1} = 0$$
$$5A^{-1} = 4I - A$$
$$A^{-1} = \frac{1}{5}(4I - A)$$

Step 4: Compute

$$4I - A = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$$
$$A^{-1} = \frac{1}{5}\begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$$

Answer (i):

$$\boxed{A^{-1} = \frac{1}{5}\begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}}$$

(ii) $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$

Step 1: Characteristic polynomial

$$\det(A - \lambda I) = \left| \begin{matrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{matrix} \right|$$

Expand along second row:

$$(1-\lambda)\left| \begin{matrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{matrix} \right|$$
$$= (1-\lambda)[(2-\lambda)^2 - 1]$$
$$= (1-\lambda)[\lambda^2 - 4\lambda + 4 - 1]$$
$$= (1-\lambda)(\lambda^2 - 4\lambda + 3)$$
$$= (1-\lambda)(\lambda - 1)(\lambda - 3)$$
$$= -(\lambda - 1)^2(\lambda - 3)$$

So $p(\lambda) = -(\lambda^3 - 5\lambda^2 + 7\lambda - 3)$, i.e.,

$$\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$$

Thus:

$$A^3 - 5A^2 + 7A - 3I = 0$$

Step 2: Express for A^{-1}
Multiply by A^{-1} (note A is invertible since det $\neq 0$):

$$A^2 - 5A + 7I - 3A^{-1} = 0$$
$$3A^{-1} = A^2 - 5A + 7I$$
$$A^{-1} = \frac{1}{3}(A^2 - 5A + 7I)$$

Step 3: Compute A^2

$$A^2 = A \cdot A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4+0+1 & 2+1+1 & 2+0+2 \\ 0+0+0 & 0+1+0 & 0+0+0 \\ 2+0+2 & 1+1+2 & 1+0+4 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$$

Step 4: Compute $A^2 - 5A + 7I$

$$A^2 - 5A + 7I = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} - 5\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

First, $5A = \begin{bmatrix} 10 & 5 & 5 \\ 0 & 5 & 0 \\ 5 & 5 & 10 \end{bmatrix}$

$$7I = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

Now:

$$A^2 - 5A + 7I = \begin{bmatrix} 5-10+7 & 4-5+0 & 4-5+0 \\ 0-0+0 & 1-5+7 & 0-0+0 \\ 4-5+0 & 4-5+0 & 5-10+7 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & 0 \\ -1 & -1 & 2 \end{bmatrix}$$

Step 5: Multiply by $\frac{1}{3}$

$$A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & 0 \\ -1 & -1 & 2 \end{bmatrix}$$

Answer (ii):

$$\boxed{A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & 0 \\ -1 & -1 & 2 \end{bmatrix}}$$

Final answers for Q2:

(i) $A^{-1} = \frac{1}{5}\begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$

(ii) $A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & 0 \\ -1 & -1 & 2 \end{bmatrix}$

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3. Find A^3 without matrix multiplication

Given $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

First, find characteristic polynomial:

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & 1 \\ 1 & -1-\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix}$$

Expand along first row:

$$(1-\lambda)[(-1-\lambda)(-\lambda) - 1] - 0 + 1[1 \cdot 1 - 0 \cdot (-1-\lambda)]$$

= $(1-\lambda)[\lambda+\lambda^2-1]+1$
= $(1-\lambda)(\lambda^2+\lambda-1)+1$

Compute: $(1-\lambda)(\lambda^2+\lambda-1)=\lambda^2+\lambda-1-\lambda^3-\lambda^2+\lambda=-\lambda^3+2\lambda-1$
Add 1: $-\lambda^3+2\lambda$

So $p(\lambda)=-\lambda^3+2\lambda=-\lambda(\lambda^2-2)$

Thus $A^3-2A=0\Rightarrow A^3=2A$.

So $A^3=2\begin{bmatrix}1&0&1\\1&-1&1\\0&1&0\end{bmatrix}=\begin{bmatrix}2&0&2\\2&-2&2\\0&2&0\end{bmatrix}$

Answer for Q3:

$$A^3=\begin{bmatrix}2&0&2\\2&-2&2\\0&2&0\end{bmatrix}$$

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4. Show that the following matrix is not diagonalizable

Given:

$$A=\begin{bmatrix}1&0&0\\1&2&0\\-3&5&2\end{bmatrix}$$

Step 1: Find eigenvalues

Matrix A is **lower triangular**.
For triangular matrices, the eigenvalues are the diagonal entries.

Thus:

$$\lambda_1=1,\lambda_2=2,\lambda_3=2$$

So eigenvalues are $\lambda=1$ (algebraic multiplicity 1) and $\lambda=2$ (algebraic multiplicity 2).

Step 2: Find eigenvectors

- **For $\lambda=1$:**
  Solve $(A-I)\mathbf{v}=0$:

$$A-I=\begin{bmatrix}0&0&0\\1&1&0\\-3&5&1\end{bmatrix}$$

Row reduce:
$R_1\leftrightarrow R_2$:

$$\begin{bmatrix}1&1&0\\0&0&0\\-3&5&1\end{bmatrix}$$

$R_3 \leftarrow R_3 + 3R_1$:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 8 & 1 \end{bmatrix}$$

From R_3: $8v_2 + v_3 = 0 \Rightarrow v_3 = -8v_2$
From R_1: $v_1 + v_2 = 0 \Rightarrow v_1 = -v_2$
Let $v_2 = t$, then $v_1 = -t$, $v_3 = -8t$.

So eigenvector for $\lambda = 1$:

$$\mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \\ -8 \end{pmatrix} \text{(or any scalar multiple)}$$

Dimension of eigenspace = 1 ✓

- **For** $\lambda = 2$ (multiplicity 2):
  Solve $(A - 2I)\mathbf{v} = 0$:

$$A - 2I = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ -3 & 5 & 0 \end{bmatrix}$$

Row reduce:
From R_1: $-v_1 = 0 \Rightarrow v_1 = 0$
From R_2: $v_1 = 0$ (same)
From R_3: $-3v_1 + 5v_2 = 0 \Rightarrow 5v_2 = 0 \Rightarrow v_2 = 0$

Thus $v_1 = 0$, $v_2 = 0$, v_3 is free.

So eigenvectors are of the form:

$$\mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ t \end{pmatrix}$$

That is, all eigenvectors for $\lambda = 2$ are scalar multiples of

$$\mathbf{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Dimension of eigenspace = 1.

Step 3: Diagonalizability check

A matrix is diagonalizable if and only if, for each eigenvalue, the **geometric multiplicity** (dimension of eigenspace) equals its **algebraic multiplicity**.

Here:

- $\lambda = 1$: algebraic multiplicity = 1, geometric multiplicity = 1 ✓

- $\lambda = 2$: algebraic multiplicity = 2, geometric multiplicity = 1 ✗

Since for $\lambda = 2$ the geometric multiplicity (1) is less than the algebraic multiplicity (2), the matrix **is not diagonalizable**.

Final Answer:

$$\boxed{\text{Matrix } A \text{ is not diagonalizable because eigenvalue } \lambda = 2 \text{ has algebraic multiplicity 2 but geometric multiplicity 1.}}$$

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5. Diagonalize given matrix and find A^{13}

Given $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$

Find eigenvalues:

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{vmatrix}$$

Expand along first row:

$$-\lambda[(2-\lambda)(3-\lambda) - 0] - 0 + (-2)[1\cdot 0 - 1\cdot(2-\lambda)]$$

$= -\lambda[(2-\lambda)(3-\lambda)] - 2[0 - (2-\lambda)]$
$= -\lambda[\lambda^2 - 5\lambda + 6] + 4 - 2\lambda$
$= -\lambda^3 + 5\lambda^2 - 6\lambda + 4 - 2\lambda$
$= -\lambda^3 + 5\lambda^2 - 8\lambda + 4$

Set to zero: $\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$
Test λ=1: 1 - 5 + 8 - 4 = 0 ⇒ factor $(\lambda - 1)$.
Divide: $(\lambda - 1)(\lambda^2 - 4\lambda + 4) = (\lambda - 1)(\lambda - 2)^2$

Eigenvalues: λ=1, λ=2 (multiplicity 2).

Eigenvectors:

- For λ=1: Solve $(A - I)v = 0 \rightarrow$ get $v_1 = (2, -1, -1)^T$
- For λ=2: Solve $(A - 2I)v = 0 \rightarrow$ get one eigenvector $v_2 = (0,1,0)^T$, and generalized maybe, but for diagonalization we need 2 independent eigenvectors for λ=2.

Check rank of $A - 2I$:

$$A - 2I = \begin{bmatrix} -2 & 0 & -2 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Thus $P = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix}$, $D = \text{diag}(1,2,2)$.

$A^{13} = PD^{13}P^{-1} = P \cdot \text{diag}(1, 2^{13}, 2^{13}) \cdot P^{-1}$.

Answer for Q5:

Diagonalizable, $A^{13} = P \begin{bmatrix} 1 & 0 & 0 \\ 0 & 8192 & 0 \\ 0 & 0 & 8192 \end{bmatrix} P^{-1}$.

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6. Orthogonally diagonalize real symmetric matrix

Given $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

Symmetric ⇒ orthogonally diagonalizable.

Find eigenvalues:

$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix}$

Compute:
Let's find by solving: Trace=11, sum of principal minors = (3*5 -1) + (3*3-1) + (5*3-1) = 14+8+14=36, det=?

Better to find λ: Try λ=2: det=

$\begin{vmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{vmatrix} = 0$ (rows 1 and 3 same) ⇒ λ=2 is eigenvalue.

Divide characteristic poly by $(\lambda-2)$: get $\lambda^2 - 9\lambda + 18 = (\lambda - 3)(\lambda - 6)$ ⇒ eigenvalues: 2,3,6.

Orthonormal eigenvectors:
For λ=2: Solve $(A - 2I)v = 0 \Rightarrow v_1 = (1,0,-1)^T$ normalize → $\frac{1}{\sqrt{2}}(1,0,-1)$
For λ=3: Solve $(A - 3I)v = 0 \Rightarrow v_2 = (1,1,1)^T$ normalize → $\frac{1}{\sqrt{3}}(1,1,1)$
For λ=6: Solve $(A - 6I)v = 0 \Rightarrow v_3 = (1,-2,1)^T$ normalize → $\frac{1}{\sqrt{6}}(1,-2,1)$

Thus $Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}$, $D = \text{diag}(2,3,6)$.

Answer for Q6:
Diagonal matrix $D = \text{diag}(2,3,6)$, orthogonal Q as above.