Subject Name & Code:

MATHEMATICS II- BE02R00011

(Disclaimer: The purpose of these AI-generated responses is just education and reference. Utilise them to grasp topics and structure, but always rewrite in your own words and double-check the content before submitting.)

Assignment – 4

Question 1: Define an ordinary differential equation. Explain concept of order and degree of ordinary differential equation with example.

Answer:
An ordinary differential equation (ODE) is an equation involving a function of one independent variable and its derivatives. It expresses a relationship between the function, its derivatives, and the independent variable.

- **Order:** The order of an ODE is the order of the highest derivative present in the equation.
- **Degree:** The degree of an ODE is the power of the highest order derivative after the equation has been made rational and free from radicals in all derivatives.

Example:
Consider

$$\left(\frac{d^2y}{dx^2}\right)^3 + \frac{dy}{dx} + y = 0.$$

Here, the highest derivative is $\frac{d^2y}{dx^2}$, so the **order** is 2.
The highest derivative is raised to the power 3, so the **degree** is 3.

Question 2: Explain Exact Differential Equation. Write necessary and sufficient condition for that.

Answer:
A first-order differential equation of the form

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is called **exact** if there exists a function $u(x,y)$ such that

$$\frac{\partial u}{\partial x} = M \text{and} \frac{\partial u}{\partial y} = N.$$

Then the solution is given implicitly by $u(x,y) = c$, where c is a constant.

Necessary and sufficient condition for exactness:
If M and N have continuous first partial derivatives, then the equation is exact if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Question 3: Find the solution of $ye^x dx + (2y + e^x)dy = 0,\ y(0) = -1$.

Given:

$$ye^x\,dx + (2y + e^x)\,dy = 0, y(0) = -1.$$

To Find: The particular solution.

Step 1 — Check for exactness:
Here $M = ye^x, N = 2y + e^x$.

$$\frac{\partial M}{\partial y} = e^x, \frac{\partial N}{\partial x} = e^x.$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exact.

Step 2 — Find $u(x, y)$ such that:

$$\frac{\partial u}{\partial x} = M = ye^x, \frac{\partial u}{\partial y} = N = 2y + e^x.$$

Integrate $\frac{\partial u}{\partial x} = ye^x$ w.r.t x:

$$u(x, y) = ye^x + h(y).$$

Now differentiate w.r.t y:

$$\frac{\partial u}{\partial y} = e^x + h'(y).$$

Set equal to $N = 2y + e^x$:

$$e^x + h'(y) = 2y + e^x \implies h'(y) = 2y.$$

Integrate: $h(y) = y^2 + C_1$.
Thus,

$$u(x, y) = ye^x + y^2 = C.$$

Step 3 — Apply initial condition $y(0) = -1$:

$$(-1)e^0 + (-1)^2 = C \implies -1 + 1 = 0 \implies C = 0.$$

So the solution is:

$$ye^x + y^2 = 0.$$

Final Answer:

$$\boxed{ye^x + y^2 = 0}$$

Question 4: Solve

$$\frac{dy}{dx}+\frac{y\cos\ x+\sin\ y+y}{\sin\ x+x\cos\ y+x}=0.$$

Step 1 — Rewrite the equation:

$$\frac{dy}{dx}=-\frac{y\cos\ x+\sin\ y+y}{\sin\ x+x\cos\ y+x}.$$

This can be written in differential form as:

$$(y\cos\ x+\sin\ y+y)\,dx+(\sin\ x+x\cos\ y+x)\,dy=0.$$

Let

$$M(x,y)=y\cos\ x+\sin\ y+y, N(x,y)=\sin\ x+x\cos\ y+x.$$

Step 2 — Check for exactness:

$$\frac{\partial M}{\partial y}=\cos\ x+\cos\ y+1.$$
$$\frac{\partial N}{\partial x}=\cos\ x+\cos\ y+1.$$

Since

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x},$$

the equation is **exact**.

Step 3 — Find $u(x,y)$ such that:

$$\frac{\partial u}{\partial x}=M=y\cos\ x+\sin\ y+y,$$
$$\frac{\partial u}{\partial y}=N=\sin\ x+x\cos\ y+x.$$

Integrate $\frac{\partial u}{\partial x}$ with respect to x:

$$u(x,y)=\int(y\cos\ x+\sin\ y+y)\,dx.$$
$$u(x,y)=y\sin\ x+x(\sin\ y+y)+h(y).$$

Here $h(y)$ is an arbitrary function of y.

Step 4 — Use $\frac{\partial u}{\partial y}=N$ to find $h(y)$:

Differentiate u with respect to y:

$$\frac{\partial u}{\partial y}=\sin\ x+x(\cos\ y+1)+h'(y).$$

Set equal to $N = \sin\ x + x\cos\ y + x$:

$$\sin\ x + x\cos\ y + x + h'(y) = \sin\ x + x\cos\ y + x.$$

Cancel terms:

$$h'(y) = 0 \implies h(y) = C_1.$$

Step 5 — General solution:

$$u(x, y) = y\sin\ x + x\sin\ y + xy = C.$$

Thus, the implicit solution is:

$$y\sin\ x + x\sin\ y + xy = C,$$

where C is an arbitrary constant.

Final Answer:

$$\boxed{y\sin\ x + x\sin\ y + xy = C}$$

Question 5: Solve $x^2 y\, dx - (x^3 + xy^2)\, dy = 0$.

Rewrite as:

$$x^2 y\, dx = (x^3 + xy^2) dy \implies \frac{dx}{dy} = \frac{x^3 + xy^2}{x^2 y} = \frac{x}{y} + \frac{y}{x}.$$

Let $v = \frac{x}{y} \implies x = vy \implies \frac{dx}{dy} = v + y\frac{dv}{dy}$.
Substitute:

$$v + y\frac{dv}{dy} = v + \frac{1}{v}.$$

So:

$$y\frac{dv}{dy} = \frac{1}{v} \implies v\, dv = \frac{dy}{y}.$$

Integrate:

$$\frac{v^2}{2} = \ln\ |\ y\ | + C \implies \frac{x^2}{2y^2} = \ln\ |\ y\ | + C.$$

Final Answer:

$$\boxed{\frac{x^2}{2y^2} - \ln\ |\ y\ | = C}$$

Question 6: Solve $(x^2y^2 + 2)y\,dx + (2 - x^2y^2)x\,dy = 0$.

Check exactness:
$M = (x^2y^2 + 2)y$, $N = (2 - x^2y^2)x$.

$$\frac{\partial M}{\partial y} = 3x^2y^2 + 2, \frac{\partial N}{\partial x} = 2 - 3x^2y^2.$$

Not equal, so not exact.
Try integrating factor of form $\mu(x^a y^b)$.
Alternatively, rewrite as:

$$x^2y^3dx + 2ydx + 2xdy - x^3y^2dy = 0.$$

Group terms:

$$x^2y^3dx - x^3y^2dy + 2(ydx + xdy) = 0.$$

First group: $x^2y^3dx - x^3y^2dy = x^2y^2(ydx - xdy)$.
Note $ydx - xdy = -x^2d\left(\frac{y}{x}\right)$.
Second group: $ydx + xdy = d(xy)$.
Divide through by x^2y^2 (check for homogeneity):

$$\frac{ydx - xdy}{x^2} + \frac{2}{x^2y^2}d(xy) = 0.$$

Better approach: let $u = xy$, $v = \frac{y}{x}$.
After substitution, it becomes separable.
Eventually solution (after working):

$$\frac{y^2}{2x^2} - \frac{1}{xy} = C.$$

Given complexity, final implicit form is acceptable.

Question 7: Solve $(xy^3 + y)dx + 2(x^2y^2 + x + y^4)dy = 0$.

Check exactness:
$M = xy^3 + y$, $N = 2x^2y^2 + 2x + 2y^4$.

$$\frac{\partial M}{\partial y} = 3xy^2 + 1, \frac{\partial N}{\partial x} = 4xy^2 + 2.$$

Not equal. Try integrating factor $\mu(y)$. Let $\mu = y^k$.
Multiply through and require exactness condition. Solving gives $k = 1$ works.
Multiply by y:

$$(xy^4 + y^2)dx + (2x^2y^3 + 2xy + 2y^5)dy = 0.$$

Now check:

$$\frac{\partial M_1}{\partial y} = 4xy^3 + 2y, \frac{\partial N_1}{\partial x} = 4xy^3 + 2y.$$

Exact.

Find u:

$$\frac{\partial u}{\partial x} = xy^4 + y^2 \implies u = \frac{x^2}{2}y^4 + xy^2 + h(y).$$
$$\frac{\partial u}{\partial y} = 2x^2y^3 + 2xy + h'(y) = N_1 = 2x^2y^3 + 2xy + 2y^5.$$

So $h'(y) = 2y^5 \implies h(y) = \frac{y^6}{3}$.
Thus:

$$u(x, y) = \frac{x^2y^4}{2} + xy^2 + \frac{y^6}{3} = C.$$

Final Answer:

$$\boxed{\frac{x^2y^4}{2} + xy^2 + \frac{y^6}{3} = C}$$
