Subject Name & Code:

MATHEMATICS II- BE02R00011

Assignment – 5

Question 1: Define Leibnitz's linear differential equation and Bernoulli's differential equation.

Answer:

- **Leibnitz's linear differential equation** (or simply *linear first-order ODE*) is of the form

$$\frac{dy}{dx} + P(x)y = Q(x),$$

where $P(x)$ and $Q(x)$ are functions of x alone. Its solution is obtained using an *integrating factor* $\mu(x) = e^{\int P(x)dx}$.

- **Bernoulli's differential equation** is of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n,$$

where $n \neq 0,1$. It can be reduced to a linear equation by the substitution $v = y^{1-n}$.

Question 2: Solve $(1 + y^2)\frac{dx}{dy} = \tan^{-1} y - x$.

Given:

$$(1 + y^2)\frac{dx}{dy} = \tan^{-1} y - x.$$

To Find: $x(y)$.

Rewrite in standard linear form in x:

$$\frac{dx}{dy} + \frac{x}{1 + y^2} = \frac{\tan^{-1} y}{1 + y^2}.$$

Here $P(y) = \frac{1}{1+y^2}$, $Q(y) = \frac{\tan^{-1} y}{1+y^2}$.

Integrating factor:

$$\mu(y) = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1} y}.$$

Multiply through:

$$\frac{d}{dy}\left(xe^{\tan^{-1}y}\right) = \frac{\tan^{-1}y}{1+y^2}e^{\tan^{-1}y}.$$

Integrate w.r.t y:
Let $t = \tan^{-1}y \implies dt = \frac{dy}{1+y^2}$.
RHS becomes $\int te^t dt$.

Using integration by parts:

$$\int te^t dt = e^t(t-1) + C.$$

Thus:

$$xe^{\tan^{-1}y} = e^{\tan^{-1}y}(\tan^{-1}y - 1) + C.$$
$$x = \tan^{-1}y - 1 + Ce^{-\tan^{-1}y}.$$

Final Answer:

$$\boxed{x = \tan^{-1}y - 1 + Ce^{-\mathrm{t}\quad^{-1}y}}$$

Question 3: Solve $\frac{dx}{dy} + (\cot\ y)x = \frac{1}{\cos y}$.

Given:

$$\frac{dx}{dy} + x\cot\ y = \frac{1}{\cos y}.$$

To Find: $x(y)$.

Integrating factor:

$$\mu(y) = e^{\int \cot\ y\, dy} = e^{\ln|\sin\ y|} = \sin\ y (\text{for } \sin\ y > 0).$$

Multiply:

$$\frac{d}{dy}(x\sin\ y) = \sin\ y \cdot \frac{1}{\cos y} = \tan\ y.$$

Integrate:

$$x\sin\ y = \int \tan\ y\, dy = \ln\ |\sec\ y| + C.$$
$$x\sin\ y = \ln\ |\sec\ y| + C.$$

Final Answer:

$$\boxed{x\sin\ y = \ln\ |\sec\ y| + C}$$

Question 4: Solve $y(y-x)dx + x^2 dy = 0$ where $y(1) = 4$.

Given:

$$y(y-x)dx + x^2dy = 0.$$

Rewrite as:

$$\frac{dy}{dx} = \frac{-y(y-x)}{x^2} = \frac{yx-y^2}{x^2}.$$

Let $y = vx \implies \frac{dy}{dx} = v + x\frac{dv}{dx}$.

Substitute:

$$v + x\frac{dv}{dx} = \frac{vx^2 - v^2x^2}{x^2} = v - v^2.$$

Thus:

$$x\frac{dv}{dx} = -v^2.$$

Separate:

$$\frac{dv}{v^2} = -\frac{dx}{x}.$$

Integrate:

$$-\frac{1}{v} = -\ln |x| + C_1 \implies \frac{1}{v} = \ln |x| + C.$$

Back substitute $v = y/x$:

$$\frac{x}{y} = \ln |x| + C.$$

Use initial condition $y(1) = 4$:

$$\frac{1}{4} = \ln 1 + C \implies C = \frac{1}{4}.$$

Thus:

$$\frac{x}{y} = \ln |x| + \frac{1}{4} \implies y = \frac{x}{\ln |x| + 1/4}.$$

Final Answer:

$$\boxed{y = \frac{x}{\ln |x| + \frac{1}{4}}}$$

Question 5: Solve $\frac{dy}{dx} + \frac{1}{dx} = \frac{e^2}{x^2}$.

The given equation seems misprinted. Likely it is:

$$\frac{dy}{dx} + \frac{1}{x}y = \frac{e^2}{x^2}.$$

Assume this form.

Integrating factor:

$$\mu(x) = e^{\int \frac{1}{x}dx} = e^{\ln} \quad = x.$$

Multiply:

$$x\frac{dy}{dx} + y = \frac{e^2}{x}.$$

Left side is $\frac{d}{dx}(xy)$.

Thus:

$$\frac{d}{dx}(xy) = \frac{e^2}{x}.$$

Integrate:

$$xy = e^2 \ln \mid x \mid + C.$$

Final Answer:

$$\boxed{xy = e^2 \ln \mid x \mid + C}$$

Question 6: Solve $r\cos\ \theta - \sin\ \theta \frac{dr}{d\theta} = r^2$ given $r\left(\frac{\pi}{2}\right) = 1$.

Given:

$$r\cos\ \theta - \sin\ \theta \frac{dr}{d\theta} = r^2.$$

Rewrite:

$$-\sin\ \theta \frac{dr}{d\theta} = r^2 - r\cos\ \theta.$$
$$\frac{dr}{d\theta} + \frac{r\cos\ \theta}{\sin\theta} = -\frac{r^2}{\sin\theta}.$$

This is Bernoulli form with $n = 2$.

Divide by r^2:

$$\frac{1}{r^2}\frac{dr}{d\theta} + \frac{\cos\theta}{\sin\theta} \cdot \frac{1}{r} = -\frac{1}{\sin\theta}.$$

Let $v = r^{-1} \implies \frac{dv}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}$.

Then:

$$-\frac{dv}{d\theta} + \frac{\cos\theta}{\sin\theta}v = -\frac{1}{\sin\theta}.$$
$$\frac{dv}{d\theta} - \cot\ \theta\ v = \csc\ \theta.$$

Linear in v.

Integrating factor:

$$\mu(\theta) = e^{-\int \cot\ \theta\, d\theta} = e^{-\ln|\sin\ \theta|} = \frac{1}{\sin\theta}.$$

Multiply:

$$\frac{d}{d\theta}\left(\frac{v}{\sin\theta}\right) = \csc\ \theta \cdot \frac{1}{\sin\theta} = \csc^2\theta.$$

Integrate:

$$\frac{v}{\sin\theta} = -\cot\ \theta + C.$$

Recall $v = 1/r$:

$$\frac{1}{r\sin\ \theta} = -\cot\ \theta + C.$$

Use $r(\pi/2) = 1$:
At $\theta = \pi/2$, $\sin\ \theta = 1$, $\cot\ \theta = 0$:

$$\frac{1}{1 \cdot 1} = 0 + C \implies C = 1.$$

Thus:

$$\frac{1}{r\sin\ \theta} = 1 - \cot\ \theta.$$
$$r\sin\ \theta(1 - \cot\ \theta) = 1.$$

Simplify $1 - \cot\ \theta = \frac{\sin\ \theta - \cos\ \theta}{\sin\theta}$:

$$r(\sin\ \theta - \cos\ \theta) = 1.$$

Final Answer:

$$\boxed{r = \frac{1}{\sin\ \theta - \cos\ \theta}}$$

Question 7: Solve $(1+x)\frac{dy}{dx} - \tan\ y = (1+x)^2 e^x \sec\ y$.

Given:

$$(1+x)\frac{dy}{dx} - \tan\ y = (1+x)^2 e^x \sec\ y.$$

Divide by $(1+x)$:

$$\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)e^x \sec\, y.$$

This is Bernoulli-type in y, but easier via substitution $z = \sin\, y$.

Better: multiply by $\cos y$:

$$(1+x)\cos\, y\frac{dy}{dx} - \sin\, y = (1+x)^2 e^x.$$

Let $z = \sin\, y \implies \frac{dz}{dx} = \cos\, y\frac{dy}{dx}$.

Then:

$$(1+x)\frac{dz}{dx} - z = (1+x)^2 e^x.$$

Divide by $(1+x)$:

$$\frac{dz}{dx} - \frac{z}{1+x} = (1+x)e^x.$$

Linear in z.

Integrating factor:

$$\mu(x) = e^{-\int \frac{1}{1+x}dx} = e^{-\ln|1+x|} = \frac{1}{1+x}.$$

Multiply:

$$\frac{d}{dx}\left(\frac{z}{1+x}\right) = e^x.$$

Integrate:

$$\frac{z}{1+x} = e^x + C.$$

Back-substitute $z = \sin\, y$:

$$\sin\, y = (1+x)(e^x + C).$$

Final Answer:

$$\boxed{\sin\, y = (1+x)(e^x + C)}$$

Question 8: Solve

$$\left(\frac{dy}{dx}\right)^2 + \frac{dy}{dx} = e^y.$$

Step 1 — Let $p = \frac{dy}{dx}$:

$$p^2 + p = e^y.$$

Step 2 — Differentiate with respect to x:

$$\frac{d}{dx}(p^2 + p) = \frac{d}{dx}(e^y).$$

Left:

$$\frac{d}{dx}(p^2) = 2p\frac{dp}{dx}, \frac{d}{dx}(p) = \frac{dp}{dx}.$$

So:

$$(2p + 1)\frac{dp}{dx} = e^y \frac{dy}{dx} = e^y p.$$

Step 3 — Substitute $e^y = p^2 + p$ from original equation:

$$(2p + 1)\frac{dp}{dx} = p(p^2 + p).$$
$$(2p + 1)\frac{dp}{dx} = p^2(p + 1).$$

Step 4 — Separate variables:

$$\frac{2p + 1}{p^2(p + 1)}\, dp = dx.$$

Step 5 — Partial fraction decomposition:
Let

$$\frac{2p + 1}{p^2(p + 1)} = \frac{A}{p} + \frac{B}{p^2} + \frac{C}{p + 1}.$$

Multiply through by $p^2(p + 1)$:

$$2p + 1 = Ap(p + 1) + B(p + 1) + Cp^2.$$

Expand:

$$2p + 1 = A(p^2 + p) + Bp + B + Cp^2.$$
$$2p + 1 = (A + C)p^2 + (A + B)p + B.$$

Compare coefficients:

1. $p^2: A + C = 0$
2. $p^1: A + B = 2$
3. $p^0: B = 1$

From $B = 1, A + 1 = 2 \Rightarrow A = 1.$
Then $A + C = 0 \Rightarrow C = -1.$

So:

$$\frac{2p + 1}{p^2(p + 1)} = \frac{1}{p} + \frac{1}{p^2} - \frac{1}{p + 1}.$$

Step 6 — Integrate:

$$\int \left(\frac{1}{p}+\frac{1}{p^2}-\frac{1}{p+1}\right) dp = \int dx.$$

$$\ln \mid p \mid -\frac{1}{p} - \ln \mid p+1 \mid = x + C_1.$$

$$\ln \mid \frac{p}{p+1} \mid -\frac{1}{p} = x + C_1.$$

Step 7 — Relate p and y using $p^2 + p = e^y$:

From $p^2 + p - e^y = 0$, solving for p:

$$p = \frac{-1 \pm \sqrt{1+4e^y}}{2}.$$

Also $\frac{p}{p+1} = \frac{p}{e^y/p} = \frac{p^2}{e^y}$ (using $p + 1 = e^y/p$ from $p^2 + p = e^y \Rightarrow p(p+1) = e^y$, so $p + 1 = e^y/p$).

Actually easier: from $p^2 + p = e^y$, we have $p + 1 = e^y/p$, so

$$\frac{p}{p+1} = \frac{p}{e^y/p} = \frac{p^2}{e^y}.$$

Then:

$$\ln \left(\frac{p^2}{e^y}\right) - \frac{1}{p} = x + C_1.$$

$$\ln (p^2) - y - \frac{1}{p} = x + C_1.$$

$$2\ln \mid p \mid - y - \frac{1}{p} = x + C_1.$$

But $p^2 + p = e^y \Rightarrow y = \ln (p^2 + p)$.

Substitute $y = \ln (p(p+1))$:

$$2\ln \mid p \mid - \ln (p(p+1)) - \frac{1}{p} = x + C_1.$$

$$2\ln \mid p \mid - \ln p - \ln \mid p+1 \mid - \frac{1}{p} = x + C_1.$$

$$\ln \mid p \mid - \ln \mid p+1 \mid - \frac{1}{p} = x + C_1.$$

This matches earlier equation exactly (consistent). So one form of the solution is the relation between p and x:

$$\ln \mid \frac{p}{p+1} \mid - \frac{1}{p} = x + C.$$

And y is given by $y = \ln (p^2 + p)$, or $e^y = p^2 + p$.

Step 8 — General solution in parametric form (p as parameter):

$$\begin{cases} x = \ln \mid \dfrac{p}{p+1} \mid - \dfrac{1}{p} + C, \\ y = \ln (p^2 + p). \end{cases}$$

Final Answer:

$$\boxed{x = \ln\ |\frac{p}{p+1}| - \frac{1}{p} + C, y = \ln\ (p^2 + p)}$$

Question 9: Solve $p = \tan\ \left(x - \frac{p}{1+p^2}\right)$.

Given:

$$p = \tan\ \left(x - \frac{p}{1+p^2}\right), p = \frac{dy}{dx}.$$

Let $u = x - \frac{p}{1+p^2}$. Then $p = \tan\ u$.

Differentiate y w.r.t x via p and eliminate p leads to a solvable form. This is a "solvable for x" type:
From $u = x - \frac{p}{1+p^2}$ and $p = \tan\ u$,

$$x = u + \frac{\tan u}{1 + \tan^2 u} = u + \frac{\sin\ u\cos\ u}{1} = u + \frac{1}{2}\sin\ 2u.$$

Also, $dy = pdx = \tan\ u\, dx$.
From $dx = (1 + \cos\ 2u)du$.

Thus:

$$dy = \tan\ u(1 + \cos\ 2u)du = \tan\ u \cdot 2\cos^2 u\, du = 2\sin\ u\cos\ u\, du = \sin\ 2u\, du.$$

Integrate:

$$y = -\frac{1}{2}\cos\ 2u + C.$$

General solution in parametric form (u parameter):

$$x = u + \frac{1}{2}\sin\ 2u, y = -\frac{1}{2}\cos\ 2u + C.$$

Final Answer:

$$\boxed{x = u + \frac{1}{2}\sin\ 2u, y = -\frac{1}{2}\cos\ 2u + C}$$

Question 10: Solve Clairaut's Equation $y = px + f(p)$.

Given Clairaut's form:

$$y = px + f(p), p = \frac{dy}{dx}.$$

Differentiate w.r.t x:

$$p = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}.$$

Thus:

$$(x + f'(p))\frac{dp}{dx} = 0.$$

Case 1: $\frac{dp}{dx} = 0 \implies p = m$ (constant).
Then general solution:

$$y = mx + f(m),$$

a family of straight lines.

Case 2: $x + f'(p) = 0$.
This gives singular solution by eliminating p between $y = px + f(p)$ and $x = -f'(p)$.

Final Answer:
General solution: $y = Cx + f(C)$.
Singular solution: $x = -f'(p), y = px + f(p)$ with p eliminated.