Subject Name & Code:

MATHEMATICS II- BE02R00011

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Assignment – 6

1. Explain Homogeneous Linear ODEs with Constant Coefficients

A homogeneous linear ordinary differential equation (ODE) with constant coefficients is an equation of the form

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

where $a_0, a_1, \dots, a_n$ are constants and $y^{(k)}$ denotes the k-th derivative of y with respect to the independent variable. The term *homogeneous* indicates that the equation equals zero (no forcing function).
The *linear* property means that y and its derivatives appear only to the first power and are not multiplied together. Such equations are solved by assuming a solution of the form $y = e^{mx}$, which leads to the characteristic (auxiliary) equation:

$$a_n m^n + a_{n-1} m^{n-1} + \cdots + a_1 m + a_0 = 0.$$

The roots of this equation determine the general solution:

- **Real and distinct roots** $m_1, m_2, \dots$: solution is $y = C_1 e^{m_1 x} + C_2 e^{m_2 x} + \cdots$
- **Repeated real root** m of multiplicity k: terms like $e^{mx}, xe^{mx}, \dots, x^{k-1} e^{mx}$ appear.
- **Complex conjugate roots** $\alpha \pm i\beta$: terms like $e^{\alpha x}(C_1 \cos\ \beta x + C_2 \sin\ \beta x)$ appear.

These ODEs frequently arise in engineering systems, such as in circuit analysis (RLC circuits) and mechanical vibrations, where they describe natural responses without external inputs.

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2. Solve the Following ODEs

(i) $y'' - 9y = 0$, with $y(0) = 2$, $y'(0) = -1$

Given:
ODE: $y'' - 9y = 0$; ICs: $y(0) = 2$, $y'(0) = -1$

To Find:
Particular solution satisfying the initial conditions.

Solution:

1. Characteristic equation: $m^2 - 9 = 0 \Rightarrow m = \pm 3$.
2. General solution: $y(x) = C_1 e^{3x} + C_2 e^{-3x}$.
3. Apply $y(0) = 2$:
   $C_1 + C_2 = 2$.

4. Derivative: $y'(x) = 3C_1e^{3x} - 3C_2e^{-3x}$.
   Apply $y'(0) = -1$:
   $3C_1 - 3C_2 = -1 \Rightarrow C_1 - C_2 = -\frac{1}{3}$.

5. Solve simultaneously:
   Adding: $2C_1 = \frac{5}{3} \Rightarrow C_1 = \frac{5}{6}$.
   Then $C_2 = 2 - \frac{5}{6} = \frac{7}{6}$.

Final Answer:

$$\boxed{y(x) = \frac{5}{6}e^{3x} + \frac{7}{6}e^{-3x}}$$

(ii) $\frac{d^4y}{dx^4} + \frac{dy}{dx} - 2y = 0$
Characteristic equation: $m^4 + m - 2 = 0$.
By inspection, $m = 1$ is a root. Factor: $(m-1)(m^3 + m^2 + m + 2) = 0$.
Cubic $m^3 + m^2 + m + 2 = 0$ has one real root $m \approx -1.353$ and complex pair $m \approx 0.1765 \pm 1.2028i$.
General solution:

$$y(x) = C_1e^x + C_2e^{-1.353x} + e^{0.1765x}(C_3\cos(1.2028x) + C_4\sin(1.2028x))$$
$$\boxed{y(x) = C_1e^x + C_2e^{-1.353x} + e^{0.1765x}(C_3\cos(1.2028x) + C_4\sin(1.2028x))}$$

(iii) $(D-2)(D^2 + D + 1)^2y = 0, D = \frac{d}{dt}$
Characteristic equation: $(m-2)(m^2 + m + 1)^2 = 0$.
Roots: $m_1 = 2$ (real), $m_{2,3} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$ (complex, multiplicity 2).
General solution:

$$y(t) = C_1e^{2t} + e^{-t/2}\left[(C_2 + C_3t)\cos\left(\frac{\sqrt{3}}{2}t\right) + (C_4 + C_5t)\sin\left(\frac{\sqrt{3}}{2}t\right)\right]$$
$$\boxed{y(t) = C_1e^{2t} + e^{-t/2}\left[(C_2 + C_3t)\cos\left(\frac{\sqrt{3}}{2}t\right) + (C_4 + C_5t)\sin\left(\frac{\sqrt{3}}{2}t\right)\right]}$$

(iv) $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$
This is a Cauchy-Euler equation. Assume $y = x^m$.
Substitute: $x^2m(m-1)x^{m-2} + x \cdot mx^{m-1} - x^m = 0$
$\Rightarrow [m(m-1) + m - 1]x^m = 0 \Rightarrow m^2 - 1 = 0 \Rightarrow m = 1, -1$.
General solution:

$$y(x) = C_1x + \frac{C_2}{x}$$
$$\boxed{y(x) = C_1x + \frac{C_2}{x}}$$

(v) $3x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} - y = 0$
Cauchy-Euler equation. Assume $y = x^m$.
Substitute: $3x^2m(m-1)x^{m-2} - x \cdot mx^{m-1} - x^m = 0$
$\Rightarrow [3m(m-1) - m - 1]x^m = 0 \Rightarrow 3m^2 - 3m - m - 1 = 0 \Rightarrow 3m^2 - 4m - 1 = 0.$
Solve: $m = \frac{4\pm\sqrt{16}}{6} = \frac{4\pm\sqrt{28}}{6} = \frac{4\pm2\sqrt{7}}{6} = \frac{2\pm\sqrt{7}}{3}$.
General solution:

$$y(x) = C_1x^{\frac{2+\sqrt{7}}{3}} + C_2x^{\frac{2-\sqrt{7}}{3}}$$

$$\boxed{y(x) = C_1x^{\frac{2+\sqrt{7}}{3}} + C_2x^{\frac{2-\sqrt{7}}{3}}}$$

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3. $3y'' - 3y' + 2y = 0$
Characteristic: $3m^2 - 3m + 2 = 0 \Rightarrow m = \frac{3\pm\sqrt{9-24}}{6} = \frac{3\pm\sqrt{-15}}{6} = \frac{3\pm i\sqrt{15}}{6} = \frac{1}{2} \pm i\frac{\sqrt{15}}{6}$.
General solution:

$$y(x) = e^{x/2}\left(C_1\cos\left(\frac{\sqrt{15}}{6}x\right) + C_2\sin\left(\frac{\sqrt{15}}{6}x\right)\right)$$

$$\boxed{y(x) = e^{x/2}\left(C_1\cos\left(\frac{\sqrt{15}}{6}x\right) + C_2\sin\left(\frac{\sqrt{15}}{6}x\right)\right)}$$

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4. $y'' + 4y = 0$
Characteristic: $m^2 + 4 = 0 \Rightarrow m = \pm 2i$.
General solution:

$$y(x) = C_1\cos(2x) + C_2\sin(2x)$$

$$\boxed{y(x) = C_1\cos(2x) + C_2\sin(2x)}$$

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5. $y'' + y' - 6y = 0$
Characteristic: $m^2 + m - 6 = 0 \Rightarrow (m+3)(m-2) = 0 \Rightarrow m = -3,2$.
General solution:

$$y(x) = C_1e^{-3x} + C_2e^{2x}$$

$$\boxed{y(x) = C_1e^{-3x} + C_2e^{2x}}$$

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6. $y'' + 2y' - 35y = 0$
Characteristic: $m^2 + 2m - 35 = 0 \Rightarrow (m+7)(m-5) = 0 \Rightarrow m = -7,5$.
General solution:

$$y(x) = C_1e^{-7x} + C_2e^{5x}$$

$$\boxed{y(x) = C_1e^{-7x} + C_2e^{5x}}$$