Subject Name & Code:

MATHEMATICS II- BE02R00011

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Assignment – 7

1. Method of Undetermined Coefficients

(i) $y'' - 3y' + 2y = e^{3x}$
Homogeneous solution: y_h.
Characteristic: $m^2 - 3m + 2 = 0 \Rightarrow (m-1)(m-2) = 0 \Rightarrow m = 1,2$.

$$y_h = C_1 e^x + C_2 e^{2x}$$

Particular solution: Since RHS = e^{3x} is not in y_h, assume $y_p = Ae^{3x}$.
Substitute:
$y_p' = 3Ae^{3x}$, $y_p'' = 9Ae^{3x}$.
Equation: $(9A - 9A + 2A)e^{3x} = e^{3x} \Rightarrow 2A = 1 \Rightarrow A = \frac{1}{2}$.
Thus $y_p = \frac{1}{2}e^{3x}$.
General solution:

$$\boxed{y = C_1 e^x + C_2 e^{2x} + \frac{1}{2}e^{3x}}$$

(ii) $y'' + 4y = \sin 3x$
Homogeneous: $m^2 + 4 = 0 \Rightarrow m = \pm 2i$.

$$y_h = C_1 \cos 2x + C_2 \sin 2x$$

RHS = $\sin 3x$ (frequency $3 \neq 2$), so assume $y_p = A\cos 3x + B\sin 3x$.
$y_p'' = -9A\cos 3x - 9B\sin 3x$.
Substitute: $(-9A\cos 3x - 9B\sin 3x) + 4(A\cos 3x + B\sin 3x) = \sin 3x$.
Coefficient match:
$\cos 3x$: $-9A + 4A = 0 \Rightarrow -5A = 0 \Rightarrow A = 0$.
$\sin 3x$: $-9B + 4B = 1 \Rightarrow -5B = 1 \Rightarrow B = -\frac{1}{5}$.
Thus $y_p = -\frac{1}{5}\sin 3x$.
General solution:

$$\boxed{y = C_1 \cos 2x + C_2 \sin 2x - \frac{1}{5}\sin 3x}$$

(iii) $y'' + y' - 6y = -6x^3 + 3x^2 + 6x$
Homogeneous: $m^2 + m - 6 = 0 \Rightarrow (m+3)(m-2) = 0 \Rightarrow m = -3,2.$

$$y_h = C_1 e^{-3x} + C_2 e^{2x}$$

RHS is cubic polynomial, so assume $y_p = Ax^3 + Bx^2 + Cx + D$.
$y_p' = 3Ax^2 + 2Bx + C, y_p'' = 6Ax + 2B$.
Substitute into ODE:
$(6Ax + 2B) + (3Ax^2 + 2Bx + C) - 6(Ax^3 + Bx^2 + Cx + D) = -6x^3 + 3x^2 + 6x$.
Match coefficients:
x^3: $-6A = -6 \Rightarrow A = 1$.
x^2: $3A - 6B = 3 \Rightarrow 3 - 6B = 3 \Rightarrow B = 0$.
x^1: $6A + 2B - 6C = 6 \Rightarrow 6 - 6C = 6 \Rightarrow C = 0$.
Constant: $2B + C - 6D = 0 \Rightarrow 0 - 6D = 0 \Rightarrow D = 0$.
Thus $y_p = x^3$.
General solution:

$$\boxed{y = C_1 e^{-3x} + C_2 e^{2x} + x^3}$$

(iv) $y'' + 2y' - 35y = 12e^{5x} + 37\sin 5x$
Homogeneous: $m^2 + 2m - 35 = 0 \Rightarrow (m+7)(m-5) = 0 \Rightarrow m = -7,5.$

$$y_h = C_1 e^{-7x} + C_2 e^{5x}$$

RHS has e^{5x} (matches $m = 5$) and $\sin 5x$.
For $12e^{5x}$: Since e^{5x} is in y_h, multiply by x: assume $y_{p1} = Axe^{5x}$.
For $37\sin 5x$: assume $y_{p2} = B\cos 5x + C\sin 5x$.
So $y_p = Axe^{5x} + B\cos 5x + C\sin 5x$.
Compute y_p' and y_p'', substitute, match coefficients:
For e^{5x} term: $(2A \cdot 5 + 2A)e^{5x}$? Let's compute carefully:

$$y_{p1} = Axe^{5x}$$
$$y_{p1}' = Ae^{5x} + 5Axe^{5x}$$
$$y_{p1}'' = 5Ae^{5x} + 5Ae^{5x} + 25Axe^{5x} = 10Ae^{5x} + 25Axe^{5x}$$

Plug into homogeneous operator $L = D^2 + 2D - 35$:

$$L[y_{p1}] = [10Ae^{5x} + 25Axe^{5x}] + 2[Ae^{5x} + 5Axe^{5x}] - 35[Axe^{5x}]$$

Coefficient of xe^{5x}: $25A + 10A - 35A = 0$ (as expected).
Coefficient of e^{5x}: $10A + 2A = 12A$.
We need $12Ae^{5x} = 12e^{5x} \Rightarrow A = 1$.
Thus $y_{p1} = xe^{5x}$.

For y_{p2}: $y_{p2} = B\cos 5x + C\sin 5x$

$$y_{p2}' = -5B\sin 5x + 5C\cos 5x$$
$$y_{p2}'' = -25B\cos 5x - 25C\sin 5x$$

Plug into L:

$$L[y_{p2}] = (-25B\cos 5x - 25C\sin 5x) + 2(-5B\sin 5x + 5C\cos 5x) - 35(B\cos 5x + C\sin 5x)$$

Combine:
$\cos 5x$: $-25B + 10C - 35B = -60B + 10C$
$\sin 5x$: $-25C - 10B - 35C = -10B - 60C$
Set equal to $0\cos 5x + 37\sin 5x$:
System:

$$-60B + 10C = 0$$

$$-10B - 60C = 37$$

From first: $C = 6B$.
Second: $-10B - 360B = 37 \Rightarrow -370B = 37 \Rightarrow B = -0.1$, then $C = -0.6$.
Thus $y_{p2} = -0.1\cos 5x - 0.6\sin 5x$.

General solution:

$$\boxed{y = C_1e^{-7x} + C_2e^{5x} + xe^{5x} - 0.1\cos 5x - 0.6\sin 5x}$$

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2. Variation of Parameters

(i) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x\log x$
Homogeneous: $m^2 - 2m + 1 = 0 \Rightarrow (m-1)^2 = 0 \Rightarrow m = 1$ (double).

$$y_h = (C_1 + C_2x)e^x$$

Two independent solutions: $y_1 = e^x$, $y_2 = xe^x$.
Wronskian: $W = y_1y_2' - y_2y_1' = e^x(e^x + xe^x) - xe^x \cdot e^x = e^{2x}$.
Particular solution: $y_p = u_1y_1 + u_2y_2$, where

$$u_1' = -\frac{y_2 \cdot f(x)}{W}, u_2' = \frac{y_1 \cdot f(x)}{W}, f(x) = e^x\log x.$$

$$u_1' = -\frac{xe^x \cdot e^x\log x}{e^{2x}} = -x\log x$$
$$u_2' = \frac{e^x \cdot e^x\log x}{e^{2x}} = \log x$$

Integrate:

$$u_1 = -\int x\log x\,dx = -\left(\frac{x^2}{2}\log x - \frac{x^2}{4}\right)$$
$$u_2 = \int \log x\,dx = x\log x - x$$

Thus

$$y_p = e^x\left[-\frac{x^2}{2}\log x + \frac{x^2}{4}\right] + xe^x[x\log x - x]$$

Simplify:

$$y_p = e^x\left[-\frac{x^2}{2}\log x + \frac{x^2}{4} + x^2\log x - x^2\right]$$
$$= e^x\left[\frac{x^2}{2}\log x - \frac{3x^2}{4}\right]$$

General solution:

$$\boxed{y = (C_1 + C_2x)e^x + e^x\left(\frac{x^2}{2}\log x - \frac{3x^2}{4}\right)}$$

(ii) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = e^x\tan x$
Homogeneous: $m^2 - 2m + 2 = 0 \Rightarrow m = 1 \pm i$.

$$y_h = e^x(C_1\cos x + C_2\sin x)$$

Solutions: $y_1 = e^x\cos x, y_2 = e^x\sin x$.
Wronskian:

$$W = y_1y_2' - y_2y_1'$$

$y_1' = e^x(\cos x - \sin x), y_2' = e^x(\sin x + \cos x)$

$$W = e^x\cos x \cdot e^x(\sin x + \cos x) - e^x\sin x \cdot e^x(\cos x - \sin x)$$
$$= e^{2x}[\cos x\sin x + \cos^2 x - \sin x\cos x + \sin^2 x]$$
$$= e^{2x}$$

$f(x) = e^x\tan x$.

$$u_1' = -\frac{y_2 f}{W} = -\frac{e^x\sin x \cdot e^x\tan x}{e^{2x}} = -\sin x\tan x = -\frac{\sin^2 x}{\cos x}$$
$$u_2' = \frac{y_1 f}{W} = \frac{e^x\cos x \cdot e^x\tan x}{e^{2x}} = \cos x\tan x = \sin x$$

Integrate:

$$u_1 = -\int \frac{\sin^2 x}{\cos x}dx = -\int \frac{1-\cos^2 x}{\cos x}dx = -\int (\sec x - \cos x)dx = -\ln |\sec x + \tan x| + \sin x$$
$$u_2 = \int \sin x\, dx = -\cos x$$

Thus $y_p = u_1y_1 + u_2y_2 = e^x\cos x[-\ln |\sec x + \tan x| + \sin x] + (-\cos x)e^x\sin x$
Simplify: $y_p = -e^x\cos x\ln |\sec x + \tan x| + e^x\cos x\sin x - e^x\sin x\cos x$
Cancels: $y_p = -e^x\cos x\ln |\sec x + \tan x|$.
General solution:

$$\boxed{y = e^x(C_1\cos x + C_2\sin x) - e^x\cos x\ln |\sec x + \tan x|}$$

(iii) $(D^2 + 1)y = \frac{1}{1+\sin x}$
Homogeneous: $m^2 + 1 = 0 \Rightarrow m = \pm i$.

$$y_h = C_1\cos x + C_2\sin x$$

$y_1 = \cos x, y_2 = \sin x$.
Wronskian: $W = \cos x \cdot \cos x - \sin x \cdot (-\sin x) = \cos^2 x + \sin^2 x = 1$.
$f(x) = \frac{1}{1+\text{si}\quad x}$.

$$u_1' = -\frac{y_2 f}{W} = -\frac{\sin x}{1+\sin x}$$
$$u_2' = \frac{y_1 f}{W} = \frac{\cos x}{1+\sin x}$$

Integrate:

$$u_1 = -\int \frac{\sin x}{1+\sin x}dx = -\int \left(1 - \frac{1}{1+\sin x}\right)dx$$
$$= -x + \int \frac{dx}{1+\sin x}$$

Let $I = \int \frac{dx}{1+\sin x}$. Multiply numerator and denominator by $1 - \sin x$:

$$I = \int \frac{1-\sin x}{\cos^2 x}dx = \int (\sec^2 x - \sec x\tan x)dx = \tan x - \sec x$$

Thus $u_1 = -x + \tan x - \sec x$.

$u_2 = \int \frac{\text{co}}{1+\sin x}dx = \ln |1 + \sin x|$.

$$y_p = u_1\cos x + u_2\sin x$$
$$= \cos x(-x + \tan x - \sec x) + \sin x\ln |1 + \sin x|$$
$$= -x\cos x + \sin x - 1 + \sin x\ln |1 + \sin x|$$

General solution:

$$\boxed{y = C_1\cos\, x + C_2\sin\, x - x\cos\, x + \sin\, x - 1 + \sin\, x \ln\, |1 + \sin\, x|}$$

(iv) $\frac{d^2y}{dx^2} + y = \sin\, x$

Homogeneous: $m^2 + 1 = 0 \Rightarrow m = \pm i$.

$y_h = C_1\cos\, x + C_2\sin\, x$.

Here $\sin x$ is in y_h (resonance). Use variation of parameters:

$y_1 = \cos\, x, y_2 = \sin\, x, W = 1, f(x) = \sin\, x$.

$$u_1' = -\frac{\sin\, x \cdot \sin\, x}{1} = -\sin^2 x$$
$$u_2' = \frac{\cos\, x \cdot \sin\, x}{1} = \frac{1}{2}\sin\, 2x$$

Integrate:

$$u_1 = -\int \sin^2 x\, dx = -\int \frac{1 - \cos\, 2x}{2} dx = -\frac{x}{2} + \frac{\sin\, 2x}{4}$$
$$u_2 = \int \frac{1}{2}\sin\, 2x\, dx = -\frac{1}{4}\cos\, 2x$$
$$y_p = u_1\cos\, x + u_2\sin\, x$$
$$= \left(-\frac{x}{2} + \frac{\sin\, 2x}{4}\right)\cos\, x + \left(-\frac{1}{4}\cos\, 2x\right)\sin\, x$$

Use $\sin\, 2x = 2\sin\, x\cos\, x$:

$$\frac{\sin\, 2x}{4}\cos\, x = \frac{1}{2}\sin\, x\cos^2 x$$

Also $-\frac{1}{4}\cos\, 2x\sin\, x = -\frac{1}{4}(1 - 2\sin^2 x)\sin\, x = -\frac{1}{4}\sin\, x + \frac{1}{2}\sin^3 x$

Combine: $y_p = -\frac{x}{2}\cos\, x + \frac{1}{2}\sin\, x\cos^2 x - \frac{1}{4}\sin\, x + \frac{1}{2}\sin^3 x$

But $\cos^2 x = 1 - \sin^2 x$:

$$\frac{1}{2}\sin\, x(1 - \sin^2 x) - \frac{1}{4}\sin\, x + \frac{1}{2}\sin^3 x = \frac{1}{2}\sin\, x - \frac{1}{2}\sin^3 x - \frac{1}{4}\sin\, x + \frac{1}{2}\sin^3 x = \frac{1}{4}\sin\, x$$

So $y_p = -\frac{x}{2}\cos\, x + \frac{1}{4}\sin\, x$.

General solution:

$$\boxed{y = C_1\cos\, x + C_2\sin\, x - \frac{x}{2}\cos\, x + \frac{1}{4}\sin\, x}$$

(Note: $\frac{1}{4}\sin\, x$ can be absorbed into $C_2\sin\, x$, so simpler form: $y = C_1\cos\, x + C_2\sin\, x - \frac{x}{2}\cos\, x$.)

(v) $\frac{d^2y}{dx^2} + a^2y = \sec\,(ax)$

Homogeneous: $m^2 + a^2 = 0 \Rightarrow m = \pm ai$.

$y_h = C_1\cos\,(ax) + C_2\sin\,(ax)$.

$y_1 = \cos\,(ax), y_2 = \sin\,(ax), W = a$.

$f(x) = \sec\,(ax)$.

$$u_1' = -\frac{\sin\,(ax)\sec\,(ax)}{a} = -\frac{\tan\,(ax)}{a}$$
$$u_2' = \frac{\cos\,(ax)\sec\,(ax)}{a} = \frac{1}{a}$$

Integrate:

$$u_1 = -\frac{1}{a}\int \tan\,(ax)dx = \frac{1}{a^2}\ln\, |\cos\,(ax)|$$
$$u_2 = \frac{x}{a}$$

$$y_p = \frac{1}{a^2}\cos(ax)\ln\ |\cos(ax)| + \frac{x}{a}\sin(ax)$$

General solution:

$$\boxed{y = C_1\cos(ax) + C_2\sin(ax) + \frac{1}{a^2}\cos(ax)\ln\ |\cos(ax)| + \frac{x}{a}\sin(ax)}$$