Subject Name & Code:

MATHEMATICS II- BE02R00011

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Assignment – 8

1. Explain Euler–Cauchy Differential Equation

An Euler–Cauchy (or equidimensional) differential equation is a linear ODE of the form

$$a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \cdots + a_1 x y' + a_0 y = f(x),$$

where the coefficients are constants multiplied by powers of x matching the order of the derivative.
The key property is that the substitution $x = e^t$ (or assuming a trial solution $y = x^m$) transforms it into a linear ODE with constant coefficients in the variable t.
For the second-order case:

$$ax^2 y'' + bxy' + cy = 0,$$

we assume $y = x^m$. Substituting gives the *indicial equation*:

$$am(m-1) + bm + c = 0.$$

Roots m_1, m_2 determine the solution structure:

- **Real and distinct**: $y = C_1 x^{m_1} + C_2 x^{m_2}$
- **Repeated real root** m: $y = (C_1 + C_2 \ln\ x) x^m$
- **Complex conjugates** $m = \alpha \pm i\beta$: $y = x^\alpha [C_1 \cos(\beta \ln\ x) + C_2 \sin(\beta \ln\ x)]$

These equations appear in engineering problems with radial or scale-invariant symmetry, e.g., in potential theory, heat conduction in cylinders, or stress analysis.

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2. Solve $x^2y'' - 3xy' + 4y = 0$, with $y(0) = 1$, $y'(1) = 3$

Given: Euler–Cauchy equation, homogeneous.
Assume $y = x^m$. Then $y' = mx^{m-1}$, $y'' = m(m-1)x^{m-2}$.
Substitute:

$$x^2[m(m-1)x^{m-2}] - 3x[mx^{m-1}] + 4x^m = 0$$
$$[m(m-1) - 3m + 4]x^m = 0$$

Indicial equation: $m^2 - 4m + 4 = 0 \Rightarrow (m-2)^2 = 0 \Rightarrow m = 2$ (double root).

General solution for repeated root:

$$y = (C_1 + C_2 \ln\ x)x^2.$$

Apply boundary conditions:

- $y(0) = 1$ is problematic because $x = 0$ is singular point
  $y(1) = C_1 = 1$.
- $y'(x) = C_2 x^2 \cdot \frac{1}{x} + 2(C_1 + C_2 \ln x)x = C_2 x + 2(C_1 + C_2 \ln x)x$.
  $y'(1) = C_2 + 2C_1 = 3$. With $C_1 = 1$, $C_2 + 2 = 3 \Rightarrow C_2 = 1$.

Thus:

$$\boxed{y = (1 + \ln x)x^2}$$

(Assuming corrected BC: $y(1) = 1, y'(1) = 3$.)

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3. Solve $x^2 y'' - 2xy' + 2y = x^3 \cos x$

Homogeneous part: $x^2 y_h'' - 2x y_h' + 2y_h = 0$.
Assume $y_h = x^m$: indicial equation $m(m-1) - 2m + 2 = 0 \Rightarrow m^2 - 3m + 2 = 0 \Rightarrow m = 1,2$.
So $y_h = C_1 x + C_2 x^2$.

Then:

$$x\frac{d}{dx} = \frac{d}{dt}, x^2\frac{d^2}{dx^2} = \frac{d^2}{dt^2} - \frac{d}{dt}.$$

Equation becomes:

$$\left(\frac{d^2 y}{dt^2} - \frac{dy}{dt}\right) - 2\frac{dy}{dt} + 2y = e^{3t}\cos(e^t).$$
$$\frac{d^2 y}{dt^2} - 3\frac{dy}{dt} + 2y = e^{3t}\cos(e^t).$$

Homogeneous solution in t: $y_h(t) = C_1 e^t + C_2 e^{2t}$, i.e., $y_h(x) = C_1 x + C_2 x^2$.

Wronskian: $W = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = 2x^2 - x^2 = x^2$.

$f(x) = x^3 \cos x / x^2 = x\cos x$ (since divide by x^2 from standard form $y'' - \frac{2}{x}y' + \frac{2}{x^2}y = x\cos x$).

Standard formula: $y_p = -y_1 \int \frac{y_2 f}{W} dx + y_2 \int \frac{y_1 f}{W} dx$.

$y_1 = x, y_2 = x^2, f = x\cos x$.

First integral: $\int \frac{y_2 f}{W} dx = \int \frac{x^2 \cdot x\cos x}{x^2} dx = \int x\cos x \, dx = x\sin x + \cos x$.

Second integral: $\int \frac{y_1 f}{W} dx = \int \frac{x \cdot x\cos x}{x^2} dx = \int \cos x \, dx = \sin x$.

Thus:

$$y_p = -x[x\sin x + \cos x] + x^2 \sin x = -x^2 \sin x - x\cos x + x^2 \sin x = -x\cos x.$$

General solution:

$$\boxed{y = C_1 x + C_2 x^2 - x\cos x}.$$

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4. Solve $x^2y'' - 4xy' + 6y = 21x - 4$

Homogeneous: $x^2y_h'' - 4xy_h' + 6y_h = 0$.
Assume $y_h = x^m$: indicial: $m(m-1) - 4m + 6 = 0 \Rightarrow m^2 - 5m + 6 = 0 \Rightarrow m = 2,3$.
So $y_h = C_1x^2 + C_2x^3$.

Particular solution: RHS is polynomial $21x - 4$. Try $y_p = Ax + B$ (since no x^2, x^3 in homogeneous? But constant and linear are not solutions of homogeneous, fine).
Substitute into original equation:

$y_p' = A, y_p'' = 0$.
LHS: $x^2(0) - 4x(A) + 6(Ax + B) = -4Ax + 6Ax + 6B = 2Ax + 6B$.
Set equal to $21x - 4$:

Coefficient of x: $2A = 21 \Rightarrow A = 10.5$.
Constant: $6B = -4 \Rightarrow B = -\frac{2}{3}$.

Thus $y_p = 10.5x - \frac{2}{3}$.

General solution:

$$\boxed{y = C_1x^2 + C_2x^3 + 10.5x - \frac{2}{3}}.$$

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5. Power series solution near $x = 0$ for $y'' + y = 0$

Assume $y = \sum_{n=0}^{\infty} a_n x^n$. Then $y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$.
Shift index: let $k = n - 2 \Rightarrow y'' = \sum_{k=0}^{\infty}(k+2)(k+1)\, a_{k+2}x^k$.

Equation: $\sum_{k=0}^{\infty}(k+2)(k+1)\, a_{k+2}x^k + \sum_{k=0}^{\infty} a_k\, x^k = 0$.

Coefficient of x^k: $(k+2)(k+1)a_{k+2} + a_k = 0$.

Recurrence: $a_{k+2} = -\frac{a_k}{(k+2)(k+1)}$.

For $k = 0$: $a_2 = -\frac{a_0}{2\cdot 1}$.
$k = 1$: $a_3 = -\frac{a_1}{3\cdot 2}$.
$k = 2$: $a_4 = -\frac{a_2}{4\cdot 3} = \frac{a_0}{4!}$.
$k = 3$: $a_5 = -\frac{a_3}{5\cdot 4} = \frac{a_1}{5!}$.

Pattern:

$$a_{2n} = (-1)^n \frac{a_0}{(2n)!}, a_{2n+1} = (-1)^n \frac{a_1}{(2n+1)!}.$$

Thus:

$$y = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$

Recognize as $y = a_0\cos\, x + a_1\sin\, x$.

$$\boxed{y = a_0\cos\,x + a_1\sin\,x}.$$

6. Power series solution for $(x^2+1)y'' + xy' - xy = 0$ about ordinary point $x = 0$

Standard form: $y'' + \frac{x}{x^2+1}y' - \frac{x}{x^2+1}y = 0$. Analytic at $x = 0$.

Assume $y = \sum_{n=0}^{\infty} a_n\,x^n$. Compute:

$y' = \sum_{n=1}^{\infty} n\,a_n x^{n-1}$, $y'' = \sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}$.

Multiply original equation term-by-term:

$(x^2+1)y'' = \sum_{n=2}^{\infty} n(n-1)\,a_n x^n + \sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}$.
$xy' = \sum_{n=1}^{\infty} n\,a_n x^n$.
$-xy = -\sum_{n=0}^{\infty} a_n\,x^{n+1}$.

Shift indices to align powers of x^n:

Second term of first: set $m = n-2$: $\sum_{m=0}^{\infty}(m+2)(m+1)\,a_{m+2}x^m$.

Combine all:

For $n = 0$: from ($m = 0$) term: $2\cdot 1a_2x^0 = 2a_2$. Others? No other x^0 terms. So $2a_2 = 0 \Rightarrow a_2 = 0$.

For $n = 1$:
From y'' part: ($m = 1$) gives $6a_3x^1$.
From xy': $n = 1$ gives $1\cdot a_1x^1$.
From $-xy$: $n = 0$ gives $-a_0x^1$.
Equation: $6a_3 + a_1 - a_0 = 0 \Rightarrow a_3 = \frac{a_0 - a_1}{6}$.

For $n \geq 2$ general recurrence:
Coefficient of x^n from:
$(x^2+1)y''$: $n(n-1)a_n + (n+2)(n+1)a_{n+2}$.
xy': na_n.
$-xy$: $-a_{n-1}$.

Sum: $[n(n-1)a_n + (n+2)(n+1)a_{n+2}] + na_n - a_{n-1} = 0$.
Simplify: $n^2a_n + (n+2)(n+1)a_{n+2} - a_{n-1} = 0$.
Thus:

$$a_{n+2} = \frac{a_{n-1} - n^2a_n}{(n+2)(n+1)}, n \geq 2.$$

We have a_0, a_1 arbitrary, $a_2 = 0$, $a_3 = (a_0 - a_1)/6$.

For $n = 2$: $a_4 = \frac{a_1 - 4a_2}{12} = \frac{a_1}{12}$.
$n = 3$: $a_5 = \frac{a_2 - 9a_3}{20} = \frac{-9(a_0 - a_1)/6}{20} = -\frac{3(a_0 - a_1)}{40}$.

Series continues. General solution:

$$\boxed{y = a_0\left[1 + \frac{x^3}{6} - \frac{3x^5}{40} + \cdots\right] + a_1\left[x - \frac{x^3}{6} + \frac{x^4}{12} + \cdots\right]}.$$

7. Power series solution near $x = 0$ for $y'' + xy = 0$

Assume $y = \sum_{n=0}^{\infty} a_n x^n$.
$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k$.
$xy = \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{k=1}^{\infty} a_{k-1} x^k$.

Equation:
$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k + \sum_{k=1}^{\infty} a_{k-1} x^k = 0$.

For $k = 0$: $2 \cdot 1 a_2 = 0 \Rightarrow a_2 = 0$.
For $k \geq 1$: $(k+2)(k+1) a_{k+2} + a_{k-1} = 0$.

Recurrence: $a_{k+2} = -\frac{a_{k-1}}{(k+2)(k+1)}$.

Let's compute:
$k = 1$: $a_3 = -\frac{a_0}{3\cdot2} = -\frac{a_0}{6}$.
$k = 2$: $a_4 = -\frac{a_1}{4\cdot3} = -\frac{a_1}{12}$.
$k = 3$: $a_5 = -\frac{a_2}{5\cdot4} = 0$ (since $a_2 = 0$).
$k = 4$: $a_6 = -\frac{a_3}{6\cdot5} = \frac{a_0}{180}$.
$k = 5$: $a_7 = -\frac{a_4}{7\cdot6} = \frac{a_1}{504}$.
$k = 6$: $a_8 = -\frac{a_5}{8\cdot7} = 0$.

Pattern: coefficients with indices multiple of 3 are special: $a_0, a_3, a_6, \ldots$ and $a_1, a_4, a_7, \ldots$, others zero except possibly higher.

General solution:

$$y = a_0\left(1 - \frac{x^3}{6} + \frac{x^6}{180} - \cdots\right) + a_1\left(x - \frac{x^4}{12} + \frac{x^7}{504} - \cdots\right).$$

$$\boxed{y = a_0 \sum_{n=0}^{\infty} c_n x^{3n} + a_1 \sum_{n=0}^{\infty} d_n x^{3n+1}},$$

with $c_0 = 1, c_1 = -\frac{1}{6}, c_2 = \frac{1}{180}, \ldots$ and $d_0 = 1, d_1 = -\frac{1}{12}, d_2 = \frac{1}{504}, \ldots$.