Subject Name & Code:

MATHEMATICS II- BE02R00011

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Assignment – 9

1. Evaluate the Limit if it exists

(i) $\lim_{z\to 0} \frac{z}{|z|}$

Let $z = x + iy$, with $|z| = \sqrt{x^2 + y^2}$.
Then

$$\frac{z}{|z|} = \frac{x + iy}{\sqrt{x^2 + y^2}}.$$

We examine the limit along two different paths.

- **Path 1:** Along real axis ($y = 0$)
  $z = x, |z| = |x|$.
  For $x > 0, \frac{z}{|z|} = \frac{x}{x} = 1$.
  For $x < 0, \frac{z}{|z|} = \frac{x}{|x|} = -1$.
  Hence along real axis, the value depends on direction; no unique limit.

- **Path 2:** Along imaginary axis ($x = 0$)
  $z = iy, |z| = |y|$.
  For $y > 0, \frac{z}{|z|} = \frac{iy}{y} = i$.
  For $y < 0, \frac{z}{|z|} = \frac{iy}{|y|} = -i$.

Since different paths give different limiting values (1, -1, i, -i), **the limit does not exist**.

$$\boxed{\text{Limit does not exist}}$$

(ii) $\lim_{z\to i} \frac{z-i}{z^2+1}$

Given: $z^2 + 1 = (z - i)(z + i)$.
Thus

$$\frac{z - i}{z^2 + 1} = \frac{z - i}{(z - i)(z + i)}.$$

For $z \neq i$, this simplifies to

$$\frac{1}{z + i}.$$

Now take limit $z \to i$:

$$\lim_{z\to i}\frac{1}{z+i}=\frac{1}{i+i}=\frac{1}{2i}.$$

Rationalize:

$$\frac{1}{2i}\cdot\frac{-i}{-i}=\frac{-i}{2}.$$

$$\boxed{-\frac{i}{2}}$$

2. Discuss the continuity of the function

(i)

$$f(z)=\begin{cases}\dfrac{\text{Re }(z^2)}{|z|}, & z\neq 0\\ 0, & z=0\end{cases}$$

At $z\neq 0$, f is a rational combination of continuous functions (Re, $|\cdot|$), so f is continuous there.

Continuity at $z=0$:
We check if $\lim_{z\to 0} f(z)=f(0)=0$.

Let $z=x+iy$, then

$$z^2=x^2-y^2+2ixy, \text{Re }(z^2)=x^2-y^2, |z|=\sqrt{x^2+y^2}.$$

Thus

$$f(z)=\frac{x^2-y^2}{\sqrt{x^2+y^2}}, z\neq 0.$$

- **Approach along real axis ($y=0$):**
  $f(z)=\frac{x^2}{\sqrt{x^2}}=\frac{x^2}{|x|}$.
  For $x>0, \frac{x^2}{x}=x\to 0$.
  For $x<0, \frac{x^2}{|x|}=-x\to 0$.
  Limit along real axis is 0.
- **Approach along line $y=x$:**
  $x^2-y^2=0$, so $f(z)=0\to$ limit 0.
- **Try $y=2x$:**
  $x^2-(2x)^2=x^2-4x^2=-3x^2, \sqrt{x^2+(2x)^2}=\sqrt{5}|x|$.
  $f(z)=\frac{-3x^2}{\sqrt{5}|x|}=-\frac{3}{\sqrt{5}}|x|\to 0$ as $x\to 0$.

All linear approaches give limit 0. However, consider a parabolic path: $y=\sqrt{mx^2+x^4}$ is messy; instead test $z=re^{i\theta}$:

$$z=re^{i\theta}, z^2=r^2e^{i2\theta}, \text{Re }(z^2)=r^2\cos 2\theta, |z|=r.$$

So for $z\neq 0$,

$$f(z) = \frac{r^2 \cos\ 2\theta}{r} = r\cos\ 2\theta.$$

As $r \to 0$, $| f(z) | \leq r \to 0$ **for any fixed** θ.

Hence $\lim_{z \to 0} f(z) = 0 = f(0)$.

Conclusion: f is continuous everywhere.

$$\boxed{\text{Continuous everywhere}}$$

(ii)

$$f(z) = \begin{cases} \frac{\bar{z}}{z}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

For $z \neq 0$, $f(z) = e^{-i2\mathrm{ar}\ \ z}$ (since $\bar{z}/z = e^{-i2\theta}$ in polar form $z = re^{i\theta}$), so it is continuous except possibly at $z = 0$.

Continuity at $z = 0$**:**
We check if $\lim_{z \to 0} f(z) = f(0) = 0$.

In polar coordinates: $z = re^{i\theta}$, $\bar{z} = re^{-i\theta}$,

$$f(z) = \frac{re^{-i\theta}}{re^{i\theta}} = e^{-2i\theta}, z \neq 0.$$

This is **independent of** r but depends on θ.

Thus for any fixed direction θ, as $r \to 0$,

$$f(z) \to e^{-2i\theta}.$$

Different values of θ give different limits:

- $\theta = 0$: limit 1
- $\theta = \pi/2$: limit $e^{-i\pi} = -1$
- $\theta = \pi/4$: limit $e^{-i\pi/2} = -i$

Hence limit depends on path → **limit does not exist** as $z \to 0$.

Since $\lim_{z \to 0} f(z)$ does not exist, f is **discontinuous at** $z = 0$.

Conclusion: f is continuous for $z \neq 0$, discontinuous at $z = 0$.

$$\boxed{\text{Continuous for } z \neq 0\text{, discontinuous at } z = 0}$$

3. Show that $f(z) = \bar{z}$ is nowhere differentiable.

Let $z = x + iy$, $\bar{z} = x - iy$.
Write $f(z) = u(x, y) + iv(x, y)$ with

$$u(x, y) = x, v(x, y) = -y.$$

Compute partial derivatives:

$$u_x = 1, u_y = 0, v_x = 0, v_y = -1.$$

Check Cauchy–Riemann equations:

$$u_x = v_y \Rightarrow 1 = -1 (\text{False}).$$

Since C–R equations fail everywhere, $f(z) = \bar{z}$ is **nowhere differentiable** (and hence nowhere analytic).

$$\boxed{\text{Nowhere differentiable}}$$

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4. Show that $f(z) = |z|^2$ is differentiable only at the point $z = 0$.

Let $z = x + iy$, then $|z|^2 = x^2 + y^2$.
Write $f(z) = u(x, y) + iv(x, y)$ with

$$u(x, y) = x^2 + y^2, v(x, y) = 0.$$

Partial derivatives:

$$u_x = 2x, u_y = 2y, v_x = 0, v_y = 0.$$

Cauchy–Riemann equations:

$$u_x = v_y \Rightarrow 2x = 0 \Rightarrow x = 0, \quad u_y = -v_x \Rightarrow 2y = 0 \Rightarrow y = 0.$$

Thus C–R equations hold **only at** $z = 0 + i0 = 0$.

Check differentiability at $z = 0$ using definition:

$$f'(0) = \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0} \frac{|h|^2}{h}.$$

Let $h = re^{i\theta}$, then $|h|^2 = r^2$,

$$\frac{|h|^2}{h} = \frac{r^2}{re^{i\theta}} = re^{-i\theta}.$$

As $r \to 0$, this tends to 0 **independent of** $\boldsymbol{\theta}$. Hence $f'(0) = 0$.

Thus $f(z) = |z|^2$ is differentiable **only at** $z = 0$.

$$\boxed{\text{Differentiable only at } z = 0}$$

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5. Determine C–R equations in polar form.

Let $z = re^{i\theta}$, with $r > 0, \theta \in \mathbb{R}$.
Write complex function $f(z) = u(r,\theta) + iv(r,\theta)$, where u, v are real-valued functions of r, θ.

From $x = r\cos\,\theta$, $y = r\sin\,\theta$, we use chain rule:

$$u_x = u_r r_x + u_\theta \theta_x, \text{etc.}$$

We know:

$$r_x = \frac{x}{r} = \cos\,\theta, r_y = \sin\,\theta, \theta_x = -\frac{y}{r^2} = -\frac{\sin\theta}{r}, \theta_y = \frac{x}{r^2} = \frac{\cos\theta}{r}.$$

So:

$$u_x = u_r \cos\,\theta - u_\theta \frac{\sin\theta}{r}, u_y = u_r \sin\,\theta + u_\theta \frac{\cos\theta}{r}.$$

Similarly,

$$v_x = v_r \cos\,\theta - v_\theta \frac{\sin\theta}{r}, v_y = v_r \sin\,\theta + v_\theta \frac{\cos\theta}{r}.$$

Cauchy–Riemann equations in Cartesian: $u_x = v_y$, $u_y = -v_x$.

Substitute:

1. $u_x = v_y$ gives:

$$u_r \cos\,\theta - \frac{u_\theta \sin\,\theta}{r} = v_r \sin\,\theta + \frac{v_\theta \cos\,\theta}{r}.$$

2. $u_y = -v_x$ gives:

$$u_r \sin\,\theta + \frac{u_\theta \cos\,\theta}{r} = -\left(v_r \cos\,\theta - \frac{v_\theta \sin\,\theta}{r}\right).$$

Multiply first equation by $\cos\theta$, second by $\sin\theta$, and add:

$$u_r = \frac{1}{r} v_\theta.$$

Multiply first equation by $-\sin\,\theta$, second by $\cos\theta$, and add:

$$\frac{1}{r} u_\theta = -v_r.$$

Thus **C–R equations in polar form**:

$$\boxed{u_r = \frac{1}{r} v_\theta, \frac{1}{r} u_\theta = -v_r}$$

6. Define analytic function. State necessary and sufficient conditions for analyticity.

Analytic function definition:
A function $f(z)$ of a complex variable z is said to be **analytic** (or **holomorphic**) at a point z_0 if it is

differentiable at every point in some neighborhood of z_0.
If $f(z)$ is analytic at every point in a domain D, it is called analytic in D. An entire function is analytic everywhere in the finite complex plane.

Necessary condition for analyticity:
If $f(z) = u(x,y) + iv(x,y)$ is analytic at $z_0 = x_0 + iy_0$, then at z_0 the Cauchy–Riemann equations must hold:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Additionally, the four first partial derivatives u_x, u_y, v_x, v_y must exist in a neighborhood of (x_0, y_0).

Sufficient condition for analyticity:
Let $f(z) = u(x,y) + iv(x,y)$ be defined in some open domain D. If

1. $u(x,y)$ and $v(x,y)$ have continuous first partial derivatives in D, and
2. The Cauchy–Riemann equations are satisfied at every point in D,
   then $f(z)$ is analytic in D.

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7. Verify that $f(z) = z^2$ satisfies CR equations and find $f'(z)$.

Let $z = x + iy$, then

$$z^2 = (x + iy)^2 = (x^2 - y^2) + i(2xy).$$

Thus $u(x,y) = x^2 - y^2$, $v(x,y) = 2xy$.

Partial derivatives:

$$u_x = 2x, u_y = -2y, v_x = 2y, v_y = 2x.$$

Check Cauchy–Riemann:

$$u_x = v_y \Rightarrow 2x = 2x \quad (\text{True}), \quad u_y = -v_x \Rightarrow -2y = -(2y) \quad (\text{True}).$$

C–R equations hold for all z.

Derivative:

$$f'(z) = u_x + iv_x = 2x + i(2y) = 2(x + iy) = 2z.$$
$$\boxed{f'(z) = 2z}$$

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8. Show that $f(z) = e^z$ is entire function.

Write $z = x + iy$, then

$$e^z = e^{x+i} \quad = e^x(\cos\ y + i\sin\ y).$$

Thus

$$u(x,y) = e^x \cos\ y, v(x,y) = e^x \sin\ y.$$

Partial derivatives:

$$u_x = e^x \cos\ y, u_y = -e^x \sin\ y, v_x = e^x \sin\ y, v_y = e^x \cos\ y.$$

Check C–R:

$$u_x = v_y \;\Rightarrow\; e^x \cos y = e^x \cos y \checkmark$$

$$u_y = -v_x \;\Rightarrow\; -e^x \sin y = -(e^x \sin y) \checkmark$$

C–R equations hold for all (x, y). Also, u_x, u_y, v_x, v_y are continuous everywhere.

Thus $f(z) = e^z$ is differentiable at every z, and is therefore **entire**.

$$\boxed{\text{Entire}}$$