

**Subject Name & Code:**  
**MATHEMATICS II- BE02R00011**

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**Assignment – 11**

**1. Using C–R equations, check whether the following functions are analytic or not at any point.**

**(i)**  $f(z) = z^3$

Let  $z = x + iy$ , then

$$z^3 = (x + iy)^3 = (x^3 - 3xy^2) + i(3x^2y - y^3).$$

So  $u = x^3 - 3xy^2$ ,  $v = 3x^2y - y^3$ .

$$\begin{aligned} u_x &= 3x^2 - 3y^2, u_y = -6xy, \\ v_x &= 6xy, v_y = 3x^2 - 3y^2. \end{aligned}$$

Check C–R:

$$u_x = v_y \Rightarrow 3x^2 - 3y^2 = 3x^2 - 3y^2 \checkmark,$$

$$u_y = -v_x \Rightarrow -6xy = -6xy \checkmark.$$

C–R hold everywhere, and partials are continuous  $\rightarrow$  **analytic everywhere.**

Analytic everywhere

**(ii)**  $f(z) = z |z|$

Let  $z = re^{i\theta}$ , then  $|z| = r$ , so

$$f(z) = re^{i\theta} \cdot r = r^2 e^{i\theta}.$$

In Cartesian form:  $r^2 = x^2 + y^2$ ,  $e^{i\theta} = \cos \theta + i \sin \theta = \frac{x}{r} + i \frac{y}{r}$ , so

$$f(z) = (x^2 + y^2) \left( \frac{x + iy}{\sqrt{x^2 + y^2}} \right) = (x^2 + y^2)^{1/2} (x + iy).$$

Better: Let  $u + iv = f(z)$ . Then

$$u = x\sqrt{x^2 + y^2}, v = y\sqrt{x^2 + y^2}.$$

$$\text{Compute } u_x = \sqrt{x^2 + y^2} + \frac{x^2}{\sqrt{x^2 + y^2}}, u_y = \frac{xy}{\sqrt{x^2 + y^2}},$$

$$v_x = \frac{xy}{\sqrt{x^2 + y^2}}, v_y = \sqrt{x^2 + y^2} + \frac{y^2}{\sqrt{x^2 + y^2}}.$$

Check  $u_x = v_y$ :

LHS:  $\sqrt{r} + \frac{x^2}{\sqrt{r}}$ , RHS:  $\sqrt{r} + \frac{y^2}{\sqrt{r}}$ , equal only if  $x^2 = y^2$ , not true generally.

So C-R fail except possibly at origin.

At  $z = 0$ , check differentiability via limit:

$$\lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{h|h|}{h} = \lim_{h \rightarrow 0} |h| = 0,$$

so  $f'(0) = 0$ , differentiable at 0 only. But analyticity requires differentiability in a neighborhood → **not analytic anywhere**.

Not analytic at any point (differentiable only at 0)

(iii)  $f(z) = \sin z$

We know  $\sin z = \sin x \cosh y + i \cos x \sinh y$ .

So  $u = \sin x \cosh y$ ,  $v = \cos x \sinh y$ .

$$\begin{aligned} u_x &= \cos x \cosh y, u_y = \sin x \sinh y, \\ v_x &= -\sin x \sinh y, v_y = \cos x \cosh y. \end{aligned}$$

Check C-R:

$$u_x = v_y \checkmark, u_y = -v_x \checkmark.$$

Continuous partials → **analytic everywhere (entire)**.

Analytic everywhere

(iv)  $f(z) = \frac{1}{z}$ ,  $z \neq 0$

Write  $f(z) = \frac{x}{x^2+y^2} - i \frac{y}{x^2+y^2}$ , so

$$u = \frac{x}{x^2+y^2}, v = -\frac{y}{x^2+y^2}.$$

$$\text{Compute } u_x = \frac{y^2-x^2}{(x^2+y^2)^2}, u_y = \frac{-2x}{(x^2+y^2)^2},$$

$$v_x = \frac{2xy}{(x^2+y^2)^2}, v_y = \frac{y^2-x^2}{(x^2+y^2)^2}.$$

Check  $u_x = v_y \checkmark$  (both  $\frac{y^2-x^2}{(x^2+y^2)^2}$ ),

$$u_y = -v_x \checkmark.$$

So C-R hold for  $z \neq 0$ , partials continuous for  $z \neq 0$  → **analytic everywhere except at  $z = 0$** .

Analytic for  $z \neq 0$

(v)  $f(z) = e^x(\cos y + i \sin y)$

But  $e^x(\cos y + i \sin y) = e^{x+iy} = e^z$ .

We already know  $e^z$  is entire (analytic everywhere).

Formally:  $u = e^x \cos y$ ,  $v = e^x \sin y$ ,

$$u_x = e^x \cos y = v_y, u_y = -e^x \sin y = -v_x.$$

Continuous partials → **analytic everywhere**.

Analytic everywhere

**2. Define a harmonic function. Check whether the following functions are Harmonic. If harmonic, find its harmonic conjugate.**

**Definition:** A real-valued function  $u(x, y)$  is **harmonic** in a domain  $D$  if it has continuous second partial derivatives in  $D$  and satisfies Laplace's equation:

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

(i)  $u = x^2 - y^2 - y$

$$u_{xx} = 2, u_{yy} = -2.$$

$$\nabla^2 u = 2 + (-2) = 0 \checkmark \text{ harmonic.}$$

Find harmonic conjugate  $v$  using C-R:  $u_x = 2x = v_y, u_y = -2y - 1 = -v_x$ .

From  $v_y = 2x$ : integrate w.r.t  $y$ :

$$v = 2xy + \phi(x).$$

From  $v_x = 2y + \phi'(x)$ , and C-R  $v_x = -u_y = 2y + 1$ , so

$$2y + \phi'(x) = 2y + 1 \Rightarrow \phi'(x) = 1 \Rightarrow \phi(x) = x + C.$$

Thus  $v = 2xy + x + C$ ,  $C$  real constant.

Harmonic; conjugate  $v = 2xy + x + C$

(ii)  $u = e^x \cos y$

$$u_{xx} = e^x \cos y, u_{yy} = -e^x \cos y.$$

$$\nabla^2 u = e^x \cos y - e^x \cos y = 0 \checkmark \text{ harmonic.}$$

C-R:  $u_x = e^x \cos y = v_y, u_y = -e^x \sin y = -v_x$ .

From  $v_y = e^x \cos y$ :

$$v = e^x \sin y + \psi(x).$$

Then  $v_x = e^x \sin y + \psi'(x)$ . C-R:  $v_x = -u_y = e^x \sin y$ , so

$$e^x \sin y + \psi'(x) = e^x \sin y \Rightarrow \psi'(x) = 0 \Rightarrow \psi(x) = C.$$

Thus  $v = e^x \sin y + C$ .

Harmonic; conjugate  $v = e^x \sin y + C$

$$(iii) u = x^2 - y^2 + \frac{x}{x^2+y^2}$$

**Step 1: Verify harmonicity**

Let  $S = x^2 + y^2$ .

Then  $u = x^2 - y^2 + \frac{x}{S}$ .

Compute first partial derivatives:

$$u_x = 2x + \frac{S - 2x^2}{S^2} = 2x + \frac{y^2 - x^2}{S^2},$$

$$u_y = -2y - \frac{2xy}{S^2}.$$

Now compute second partial derivatives:

$$u_{xx} = 2 + \frac{-2x(3y^2 - x^2)}{S^3}, u_{yy} = -2 + \frac{2x(3y^2 - x^2)}{S^3}.$$

Add them:

$$\nabla^2 u = u_{xx} + u_{yy} = \left[ 2 + \frac{-2x(3y^2 - x^2)}{S^3} \right] + \left[ -2 + \frac{2x(3y^2 - x^2)}{S^3} \right] = 0.$$

Hence  $u$  is harmonic (except at  $z = 0$  where  $S = 0$ ).

**Step 2: Find harmonic conjugate  $v$**

Cauchy–Riemann equations:  $u_x = v_y, u_y = -v_x$ .

From  $u_x = v_y$ :

$$v_y = 2x + \frac{y^2 - x^2}{S^2}.$$

Integrate with respect to  $y$ :

$$v = 2xy + \int \frac{y^2 - x^2}{S^2} dy.$$

Using known derivative  $\frac{\partial}{\partial y} \left( \frac{y}{S} \right) = \frac{x^2 - y^2}{S^2}$ , we get

$$\int \frac{y^2 - x^2}{S^2} dy = -\frac{y}{S}.$$

Thus

$$v = 2xy - \frac{y}{S} + \phi(x).$$

Now use  $u_y = -v_x$ :

$$u_y = -2y - \frac{2xy}{S^2}.$$

Compute  $v_x$  from  $v = 2xy - \frac{y}{S} + \phi(x)$ :

$$v_x = 2y + \frac{2xy}{S^2} + \phi'(x).$$

Then

$$u_y = -v_x \Rightarrow -2y - \frac{2xy}{S^2} = -\left[2y + \frac{2xy}{S^2} + \phi'(x)\right].$$

Simplify:

$$-2y - \frac{2xy}{S^2} = -2y - \frac{2xy}{S^2} - \phi'(x) \Rightarrow \phi'(x) = 0.$$

Hence  $\phi(x) = C$  (a real constant).

**Final harmonic conjugate:**

$$v(x, y) = 2xy - \frac{y}{x^2 + y^2} + C$$

### 3. Determine Analytic function whose imaginary part is $e^x(x\cos y - y\sin y)$

Given  $v(x, y) = e^x(x\cos y - y\sin y)$ .

We need analytic  $f(z) = u + iv$  with  $v$  as given.

Use C-R:  $u_x = v_y$  and  $u_y = -v_x$ .

First compute:

$$v_y = e^x[-x\sin y - (\sin y + y\cos y)] = e^x[-x\sin y - \sin y - y\cos y].$$

So  $u_x = v_y = -e^x[(x+1)\sin y + y\cos y]$ .

Now  $v_x = e^x(x\cos y - y\sin y) + e^x(\cos y) = e^x[(x+1)\cos y - y\sin y]$ .

Then  $u_y = -v_x = -e^x[(x+1)\cos y - y\sin y]$ .

Integrate  $u_x$  w.r.t  $x$ :

$$u = \int -e^x[(x+1)\sin y + y\cos y] dx.$$

Let  $A(y) = \sin y$ ,  $B(y) = y\cos y$ .

Integrate  $-e^x[(x+1)A + B]$ .

Use integration by parts for  $-e^x(x+1)A$ :

$$\int -e^x(x+1)A dx = -A \int e^x(x+1) dx.$$

Let  $\int e^x(x+1) dx = e^x x + C$  (since derivative of  $e^x x$  is  $e^x x + e^x$ , indeed  $\int e^x(x+1) dx = e^x x + C$ ).

So  $-A \cdot e^x x = -x e^x \sin y$ .

And  $\int -e^x B dx = -Be^x = -ye^x \cos y$ .

Thus

$$u = -xe^x \sin y - ye^x \cos y + g(y).$$

Now use  $u_y$  from earlier:

From this  $u$ ,

$$u_y = -xe^x \cos y - e^x \cos y + ye^x \sin y - ye^x \sin y + g'(y),$$

Wait carefully:

$$u = -e^x x \sin y - e^x y \cos y + g(y).$$

$u_y = -e^x x \cos y - e^x [\cos y - y \sin y] + g'(y)$  (product rule on  $y \cos y$ : derivative of  $y \cos y$  is  $\cos y - y \sin y$ ).

So

$$u_y = -e^x x \cos y - e^x \cos y + e^x y \sin y + g'(y).$$

But earlier from C-R we had  $u_y = -e^x [(x+1) \cos y - y \sin y] = -e^x (x+1) \cos y + e^x y \sin y$ .

Comparing: both have  $-e^x x \cos y - e^x \cos y + e^x y \sin y$ , so  $g'(y) = 0 \Rightarrow g(y) = C$ , constant.

Thus  $u = e^x (-x \sin y - y \cos y) + C$ .

Hence

$$f(z) = u + iv = e^x [-x \sin y - y \cos y + i(x \cos y - y \sin y)] + C.$$

Recognize:

Group terms:  $e^x [(-y \cos y + ix \cos y) + (-x \sin y - iy \sin y)]$

Factor  $i$ :

$$= e^x [i^2 y \cos y + ix \cos y + i^2 x \sin y - iy \sin y]$$

Better: write  $f(z) = e^x [(ix - y) \cos y + (-x - iy) \sin y]$ .

But note:  $i(x + iy) = iz$ , and  $z = x + iy$ , maybe simpler:

Check if  $f(z) = iz e^z$  works:

Let  $z e^z = (x + iy) e^x (\cos y + i \sin y)$ .

Imag part of  $iz e^z = \operatorname{Re}(z e^z) = e^x (x \cos y - y \sin y)$  — yes, matches given  $v$ .

So

$$\boxed{f(z) = iz e^z + C, C \text{ real constant}}$$

#### 4. Determine Analytic function whose real part is $u = x^4 - 6x^2y^2 + y^4$ , and find $v$ .

Given  $u = x^4 - 6x^2y^2 + y^4$ .

First check harmonic:

$$\begin{aligned} u_{xx} &= 12x^2 - 12y^2, u_{yy} = -12x^2 + 12y^2. \\ \nabla^2 u &= (12x^2 - 12y^2) + (-12x^2 + 12y^2) = 0 \checkmark \text{ harmonic.} \end{aligned}$$

C-R:  $u_x = 4x^3 - 12xy^2 = v_y, u_y = -12x^2y + 4y^3 = -v_x$ .

Integrate  $v_y$  w.r.t  $y$ :

$$v = 4x^3y - 12x\frac{y^3}{3} + \phi(x) = 4x^3y - 4xy^3 + \phi(x).$$

Now  $v_x = 12x^2y - 4y^3 + \phi'(x)$ .

But from C-R:  $v_x = -u_y = 12x^2y - 4y^3$ .

$$\text{Thus } 12x^2y - 4y^3 + \phi'(x) = 12x^2y - 4y^3 \Rightarrow \phi'(x) = 0 \Rightarrow \phi(x) = C.$$

So  $v = 4x^3y - 4xy^3 + C$ .

Now  $f(z) = u + iv$ . Observe:

$$u = x^4 - 6x^2y^2 + y^4, v = 4x^3y - 4xy^3.$$

We can write  $f(z) = (x + iy)^4 = z^4$ , since  
 $z^4 = (x^4 - 6x^2y^2 + y^4) + i(4x^3y - 4xy^3)$ .

Thus

$$\boxed{f(z) = z^4 + iC, v = 4x^3y - 4xy^3 + C}$$