Subject Name & Code:

MATHEMATICS II- BE02R00011

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Assignment – 12

Question 1

Evaluate $\int_C z^2\ dz$ where C is the line joining the points (0, 0) and (4, 2).

Given:

- $z = x + iy$
- C: straight line from 0 to $4 + 2i$

To Find: Value of the contour integral $\int_C z^2\ dz$.

Formula:
For a smooth curve C parameterized by $z(t)$, $t \in [a, b]$:

$$\int_C f(z)\, dz = \int_a^b f(z(t))\ z'(t)\, dt$$

Solution:

1. Parameterize C:
   Let $x = 4t$, $y = 2t$, $t \in [0,1]$.
   Then $z(t) = 4t + i(2t) = 2t(2 + i)$.
   $z'(t) = 4 + 2i$.

2. Compute $z(t)^2$:

$$z(t)^2 = (4t + 2it)^2 = 16t^2 + 16it^2 - 4t^2 = 12t^2 + 16it^2 = 4t^2(3 + 4i)$$

3. Set up the integral:

$$\int_C z^2\ dz = \int_0^1 [4\,t^2(3 + 4i)] \cdot (4 + 2i)\, dt$$

$$= 4(3 + 4i)(4 + 2i) \int_0^1 t^2\ dt$$

Compute $(3 + 4i)(4 + 2i) = 12 + 6i + 16i + 8i^2 = 12 + 22i - 8 = 4 + 22i$

$$= 4(4 + 22i) \cdot \left[\frac{t^3}{3}\right]_0^1 = \frac{4}{3}(4 + 22i)$$

Final Answer:

$$\boxed{\frac{16}{3} + i\frac{88}{3}}$$

Question 2

Evaluate $\int_C (x^2 + ixy)\, dz$ from points (1, 1) to (2, 4) along the curve $x = t, y = t^2$.

Given:

- $x = t, y = t^2, t \in [1,2]$
- $z = x + iy = t + it^2$

To Find: $\int_C (x^2 + ixy)\, dz$.

Formula: Same as above.

Solution:

1. Parameterize: $z(t) = t + it^2, z'(t) = 1 + 2it$.
2. Integrand:

$$x^2 + ixy = t^2 + i(t)(t^2) = t^2 + it^3$$

3. Integral:

$$\int_C (x^2 + ixy)\, dz = \int_1^2 (t^2 + it^3)(1 + 2it)\, dt$$

Expand:

$$= \int_1^2 [t^2 + it^3 + 2it^3 + 2i^2 t^4]\, dt$$
$$= \int_1^2 [t^2 + 3it^3 - 2t^4]\, dt$$

Integrate term-wise:

$$= \left[\frac{t^3}{3} + 3i\frac{t^4}{4} - 2\frac{t^5}{5}\right]_1^2$$

Evaluate:

$$= \left(\frac{8}{3} + 3i \cdot 4 - \frac{64}{5}\right) - \left(\frac{1}{3} + \frac{3i}{4} - \frac{2}{5}\right)$$

Simplify real and imaginary parts separately:
Real: $\frac{8}{3} - \frac{64}{5} - \frac{1}{3} + \frac{2}{5} = \frac{7}{3} - \frac{62}{5} = \frac{35-1}{15} = -\frac{151}{15}$
Imaginary: $12i - \frac{3i}{4} = \frac{48i - 3i}{4} = \frac{45i}{4}$

Final Answer:

$$\boxed{-\frac{151}{15} + i\frac{45}{4}}$$

Question 3

Evaluate $\int_C \mathrm{Re}\,(z^2)\,dz$ where C is the boundary of the square with vertices $0, i, 1+i, 1$ in the clockwise direction.

Given:
Square vertices: $0, i, 1+i, 1$.
Clockwise direction: $0 \to i \to 1+i \to 1 \to 0$.
$\mathrm{Re}\,(z^2) = x^2 - y^2$.

To Find: Contour integral of $\mathrm{Re}\,(z^2)$.

Formula:

$$\int_C f(z)\,dz = \int_C (u\,dx - v\,dy) + i\int_C (u\,dy + v\,dx)$$

where $f(z) = u + iv$.

Solution:
Here $f(z) = \mathrm{Re}\,(z^2) = x^2 - y^2$ is **not analytic** (fails Cauchy-Riemann), so direct parameterization is needed.

Segment-wise evaluation:

1. **Segment 1:** 0 to i

$$x = 0, dx = 0, y: 0 \to 1$$

$$\int_{C_1} (x^2 - y^2)(dx + idy) = \int_0^1 (-y^2)(i\,dy) = -i\int_0^1 y^2\ dy = -i \cdot \frac{1}{3}$$

2. **Segment 2:** i to $1+i$

$$y = 1, dy = 0, x: 0 \to 1$$

$$\int_{C_2} (x^2 - 1)(dx) = \int_0^1 (x^2 - 1)\,dx = \left[\frac{x^3}{3} - x\right]_0^1 = \frac{1}{3} - 1 = -\frac{2}{3}$$

3. **Segment 3:** $1+i$ to 1

$$x = 1, dx = 0, y: 1 \to 0$$

$$\int_{C_3} (1 - y^2)(i\,dy) = i\int_1^0 (1 - y^2)\,dy = i\left[y - \frac{y^3}{3}\right]_1^0 = i(0 - (1 - \frac{1}{3})) = -i \cdot \frac{2}{3}$$

4. **Segment 4:** 1 to 0

$$y = 0, dy = 0, x: 1 \to 0$$

$$\int_{C_4} (x^2)(dx) = \int_1^0 x^2\ dx = \left[\frac{x^3}{3}\right]_1^0 = -\frac{1}{3}$$

Sum all segments:
Real parts: $0 + (-\frac{2}{3}) + 0 + (-\frac{1}{3}) = -1$
Imaginary parts: $-\frac{i}{3} + 0 + (-\frac{2i}{3}) + 0 = -i$

Final Answer:

$$\boxed{-1 - i}$$

Question 4

Evaluate $\int_C \frac{e^{-z}}{z+1}\,dz$; C is $|z| = \frac{1}{2}$.

Given:
$C: |z| = \frac{1}{2}$, center at origin, radius 0.5.
Integrand: $f(z) = \frac{e^{-z}}{z+1}$.

To Find: Contour integral value.

Formula: Cauchy's Integral Formula:
If f is analytic inside and on C, and z_0 inside C:

$$\oint_C \frac{f(z)}{z - z_0}\,dz = 2\pi i\, f(z_0)$$

Solution:
Singularity of $\frac{e^{-z}}{z+1}$ is at $z = -1$.
Check if $z = -1$ lies inside $|z| = 0.5$:
$|-1| = 1 > 0.5 \rightarrow$ singularity is **outside** the contour.
Thus, integrand is analytic **inside** C.
By Cauchy-Goursat theorem:

$$\oint_C \frac{e^{-z}}{z+1}\,dz = 0$$

Final Answer:

$$\boxed{0}$$

Question 5

State Cauchy's Integral Theorem. Evaluate $\int_C e^{-z^2}\,dz$, where C is any closed contour. Justify your Answer.

Cauchy's Integral Theorem:
If a function $f(z)$ is analytic at all points inside and on a simple closed contour C, then

$$\oint_C f(z)\,dz = 0.$$

Given: $f(z) = e^{-z^2}$, C = any closed contour.

To Find: $\oint_C e^{-z^2}\,dz$.

Solution:
e^{-z^2} is an entire function (analytic for all finite z).
By Cauchy's Integral Theorem, for any closed contour C:

$$\oint_C e^{-z^2}\,dz = 0$$

Justification:
The integrand is entire, so it satisfies the conditions of Cauchy's theorem for any closed path.

Final Answer:

$$\boxed{0}$$

Question 6
Evaluate $\oint_C \frac{z-1}{(z+1)^2(z-2)}\,dz$, where C is the circle $|z-i|=2$.

Given:
$C: |z-i|=2$, center i, radius 2.
Integrand: $f(z)=\frac{z-1}{(z+1)^2(z-2)}$.

To Find: Value of the contour integral.

Formula:
Cauchy's Integral Formula for higher-order poles:

$$\oint_C \frac{g(z)}{(z-z_0)^n}\,dz = \frac{2\pi i}{(n-1)!}\,g^{(n-1)}(z_0)$$

if $g(z)$ is analytic inside C.

Solution:
Singularities:

- $z=-1$ (double pole, from $(z+1)^2$)
- $z=2$ (simple pole)

Check if inside $C: |z-i|=2$:

- Distance from i to -1: $|-1-i| = \sqrt{(-1)^2+(-1)^2} = \sqrt{2} \approx 1.414 < 2 \rightarrow$ **inside**
- Distance from i to 2: $|2-i| = \sqrt{(2)^2+(-1)^2} = \sqrt{5} \approx 2.236 > 2 \rightarrow$ **outside**

Only $z=-1$ lies inside C.

Write integrand as:

$$\frac{z-1}{(z+1)^2(z-2)} = \frac{\frac{z-1}{z-2}}{(z+1)^2}$$

Let $g(z)=\frac{z-1}{z-2}$, analytic near $z=-1$ (since $z=2$ is outside).

For $n=2$:

$$\oint_C \frac{g(z)}{(z+1)^2}\,dz = 2\pi i\, g'(-1)$$

Compute $g(z)=\frac{z-1}{z-2}$.
Using quotient rule:

$$g'(z) = \frac{(1)(z-2)-(z-1)(1)}{(z-2)^2} = \frac{z-2-z+1}{(z-2)^2} = \frac{-1}{(z-2)^2}$$

At $z = -1$:

$$g'(-1) = \frac{-1}{(\ \ -1-2)^2} = \frac{-1}{9} = -\frac{1}{9}$$

Thus:

$$\oint_C = 2\pi i \cdot \left(\ \ -\frac{1}{9}\right) = -\frac{2\pi i}{9}$$

Final Answer:

$$\boxed{-\frac{2\pi i}{9}}$$

Question 7

Evaluate $\oint_C \frac{1}{(z^3-1)^2}\, dz$, where C is the circle $|\ z-1\ |= 1$.

Given:
C: $|\ z-1\ |= 1$, center 1, radius 1.
Integrand: $f(z) = \frac{1}{(z^3-1)^2}$.

To Find: Contour integral.

Solution:
Factor: $z^3 - 1 = (z-1)(z^2+z+1)$.
Singularities:

- $z = 1$ (double pole, since square of $(z-1)$ term)
- $z = e^{2\pi i/3}, e^{4\pi i/3}$ (roots of $z^2+z+1 = 0$, each double due to square)

Check inclusion in C:

- $z = 1$: distance from center 1 is $0 < 1 \to$ inside
- $e^{2\pi\ /3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$: distance from center 1 is $|\ -\frac{3}{2} + i\frac{\sqrt{3}}{2}\ |= \sqrt{\frac{9}{4}+\frac{3}{4}} = \sqrt{3} \approx 1.732 > 1 \to$ outside
- $e^{4\pi i/3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$: same distance $\sqrt{3} > 1 \to$ outside

Only $z = 1$ is inside C.

Write:

$$\frac{1}{(z^3-1)^2} = \frac{1}{(z-1)^2(z^2+z+1)^2}$$

Let $g(z) = \frac{1}{(z^2+z+1)^2}$, analytic near $z = 1$.

For $n = 2$:

$$\oint_C \frac{g(z)}{(z-1)^2}\, dz = 2\pi i\, g'(1)$$

Compute $g(z) = (z^2 + z + 1)^{-2}$.
Derivative:

$$g'(z) = -2(z^2 + z + 1)^{-3}(2z + 1)$$

At $z = 1$: $z^2 + z + 1 = 3$, $2z + 1 = 3$.

$$g'(1) = -2 \cdot 27^{-1} \cdot 3 = -\frac{6}{27} = -\frac{2}{9}$$

Thus:

$$\oint_C = 2\pi i \cdot \left(-\frac{2}{9}\right) = -\frac{4\pi i}{9}$$

Final Answer:

$$\boxed{-\frac{4\pi i}{9}}$$

Question 8
Evaluate $\oint_C \frac{2z+6}{z^2+4}\, dz$, where C is the circle $|\, z - i \,| = 2$.

Given:
C: $|\, z - i \,| = 2$, center i, radius 2.
Integrand: $f(z) = \frac{2z+6}{z^2+4}$.

To Find: Contour integral.

Solution:
Factor denominator: $z^2 + 4 = (z - 2i)(z + 2i)$.
Singularities at $z = \pm 2i$.

Check inclusion in C:

- Distance from center i to $2i$: $|\, 2i - i \,| = |\, i \,| = 1 < 2 \rightarrow$ inside
- Distance from center i to $-2i$: $|\, -2i - i \,| = |\, -3i \,| = 3 > 2 \rightarrow$ outside

Only $z_0 = 2i$ lies inside C.

Write integrand as:

$$\frac{2z + 6}{z^2 + 4} = \frac{2z + 6}{(z - 2i)(z + 2i)} = \frac{\frac{2z + 6}{z + 2i}}{z - 2i}$$

Let $g(z) = \frac{2z+6}{z+2i}$, analytic near $z = 2i$ (since $z = -2i$ is outside).

By Cauchy's Integral Formula for simple pole ($n = 1$):

$$\oint_C \frac{g(z)}{z - 2i}\, dz = 2\pi i\, g(2i)$$

Compute $g(2i)$:

$$g(2i) = \frac{2(2i) + 6}{2i + 2i} = \frac{4i + 6}{4i}$$

Simplify:

$$g(2i) = \frac{4i}{4i} + \frac{6}{4i} = 1 + \frac{3}{2i} = 1 - \frac{3i}{2} \text{(since } 1/i = -i\text{)}$$

Thus:

$$\oint_C = 2\pi i \cdot \left(1 - \frac{3i}{2}\right) = 2\pi i - 3\pi i^2 = 2\pi i + 3\pi$$

(since $i^2 = -1$).

Final Answer:

$$\boxed{3\pi + 2\pi i}$$