Subject Name & Code:

MATHEMATICS II- BE02R00011

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Assignment – 13

Question 1
State Cauchy's integral formula. Hence evaluate

$$\oint_C \frac{z^2}{(z-1)(z-2)}\,dz, C: |\,z\,|= 3.$$

Cauchy's Integral Formula

If $f(z)$ is analytic inside and on a simple closed contour C, and z_0 is any point inside C, then

$$\oint_C \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0).$$

For higher-order poles (order n):

$$\oint_C \frac{f(z)}{(z-z_0)^n}\,dz = \frac{2\pi i}{(n-1)!} f^{(n-1)}(z_0),$$

provided $f(z)$ is analytic inside and on C.

Given:

$$I = \oint_{|z|=3} \frac{z^2}{(z-1)(z-2)}\,dz$$

To Find: Value of I.

Solution:

Step 1: Identify singularities
The integrand has simple poles at

$$z = 1 \text{and} z = 2.$$

Step 2: Check if singularities lie inside C
$C: |z| = 3$ is a circle centered at 0 with radius 3.
Both $|1| = 1 < 3$ and $|2| = 2 < 3$, so **both poles are inside C.**

Step 3: Use partial fractions to split integrand
Write

$$\frac{z^2}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}.$$

Solving for A and B:
Multiply through by $(z-1)(z-2)$:

$$z^2 = A(z-2) + B(z-1).$$

Substitute $z = 1$:
$1 = A(-1) \implies A = -1.$
Substitute $z = 2$:
$4 = B(1) \implies B = 4.$
Thus:

$$\frac{z^2}{(z-1)(z-2)} = -\frac{1}{z-1} + \frac{4}{z-2}.$$

Step 4: Apply Cauchy's Integral Formula
Since both poles are inside C, we integrate term-wise:

$$I = \oint_C \left(-\frac{1}{z-1} + \frac{4}{z-2}\right) dz.$$

For $-\frac{1}{z-1}$, take $f(z) = -1$ (constant, analytic).
Cauchy's formula gives:

$$\oint_C \frac{-1}{z-1}\, dz = 2\pi i \cdot (-1) = -2\pi i.$$

For $\frac{4}{z-2}$, take $f(z) = 4$ (constant).

$$\oint_C \frac{4}{z-2}\, dz = 2\pi i \cdot 4 = 8\pi i.$$

Step 5: Sum results

$$I = (-2\pi i) + (8\pi i) = 6\pi i.$$

Final Answer:

$$\boxed{6\pi i}$$

Question 2

Evaluate $\oint_C \frac{z^2}{(z-3i)^2}\, dz$; C is $|z| = 5$.

Given:

$$I = \oint_{|z|=5} \frac{z^2}{(z-3i)^2}\, dz$$

To Find: Value of I.

Solution:

Step 1: Identify singularity
The integrand has a pole of order 2 at

$$z_0 = 3i.$$

Step 2: Check if inside C
$C: |z| = 5$ is centered at origin, radius 5.
$|3i| = 3 < 5 \rightarrow$ **inside**.

Step 3: Apply formula for higher-order pole
For $n = 2$:

$$\oint_C \frac{f(z)}{(z-z_0)^2}\, dz = 2\pi i\, f'(z_0)$$

where $f(z) = z^2$ is analytic everywhere.

Step 4: Compute derivative

$$f'(z) = 2z \Rightarrow f'(3i) = 2(3i) = 6i.$$

Step 5: Evaluate

$$I = 2\pi i \cdot f'(3i) = 2\pi i \cdot (6i) = 12\pi i^2.$$

Since $i^2 = -1$:

$$I = -12\pi.$$

Final Answer:

$$\boxed{-12\pi}$$

Question 3

Evaluate $\oint_C \frac{1}{(z^3-1)^2}\, dz$, where C is the circle $|\, z-1 \,|= 1$.

Given:

$$I = \oint_{|z-1|=1} \frac{dz}{(z^3-1)^2}$$

To Find: Value of I.

Solution:

Step 1: Factor denominator

$$z^3 - 1 = (z-1)(z^2+z+1).$$

So

$$\frac{1}{(z^3-1)^2} = \frac{1}{(z-1)^2(z^2+z+1)^2}.$$

Step 2: Singularities

- $z = 1$: double pole from $(z-1)^2$
- $z = e^{2\pi\,/3}, e^{4\pi i/3}$: roots of $z^2+z+1=0$, each double from the square.

Step 3: Check which lie inside C
$C\!:|\, z-1 \,|= 1$, center 1, radius 1.

- $z = 1$: distance $0 < 1 \to$ inside
- $e^{2\pi\,/3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$: distance from center 1:
- $e^{4\pi i/3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$: same distance $\sqrt{3} > 1 \to$ outside.

Only $z = 1$ is inside C.

Step 4: Use formula for double pole
Write

$$\frac{1}{(z-1)^2(z^2+z+1)^2} = \frac{g(z)}{(z-1)^2}$$

where $g(z) = \frac{1}{(z^2+z+1)^2}$, analytic near $z = 1$.

For $n = 2$:

$$I = \oint_C \frac{g(z)}{(z-1)^2}\, dz = 2\pi i\, g'(1).$$

Step 5: Compute $g'(1)$
First, $g(z) = (z^2 + z + 1)^{-2}$.
Derivative:

$$g'(z) = -2(z^2 + z + 1)^{-3}(2z + 1).$$

At $z = 1$: $z^2 + z + 1 = 3$, $2z + 1 = 3$.

$$g'(1) = -2 \cdot (27)^{-1} \cdot 3 = -\frac{6}{27} = -\frac{2}{9}.$$

Step 6: Evaluate

$$I = 2\pi i \cdot \left(-\frac{2}{9}\right) = -\frac{4\pi i}{9}.$$

Final Answer:

$$\boxed{-\frac{4\pi i}{9}}$$

Question 4
Evaluate $\oint_C \frac{2z+6}{z^2+4}\, dz$, where C is the circle $|z - i| = 2$.

Given:

$$I = \oint_{|z-i|=2} \frac{2z + 6}{z^2 + 4}\, dz$$

To Find: Value of I.

Solution:

Step 1: Factor denominator

$$z^2 + 4 = (z - 2i)(z + 2i).$$

Step 2: Singularities
Poles at $z = 2i$ and $z = -2i$.

Step 3: Check which are inside C
$C: |z - i| = 2$, center i, radius 2.

- Distance from i to $2i$: $|2i - i| = |i| = 1 < 2 \rightarrow$ inside.
- Distance from i to $-2i$: $|-2i - i| = |-3i| = 3 > 2 \rightarrow$ outside.

Only $z_0 = 2i$ is inside.

Step 4: Write integrand in Cauchy form

$$\frac{2z+6}{z^2+4} = \frac{2z+6}{(z-2i)(z+2i)} = \frac{\frac{2z+6}{z+2i}}{z-2i}.$$

Let $f(z) = \frac{2z+6}{z+2i}$, analytic near $z = 2i$ (since $z = -2i$ is outside).

Step 5: Apply Cauchy's formula

$$I = \oint_C \frac{f(z)}{z-2i}\, dz = 2\pi i\, f(2i).$$

Step 6: Compute $f(2i)$

$$f(2i) = \frac{2(2i)+6}{2i+2i} = \frac{4i+6}{4i}.$$

Simplify:

$$f(2i) = \frac{4i}{4i} + \frac{6}{4i} = 1 + \frac{3}{2i}.$$

Since $\frac{1}{i} = -i$:

$$1 + \frac{3}{2i} = 1 - \frac{3i}{2}.$$

Step 7: Evaluate

$$I = 2\pi i\left(1 - \frac{3i}{2}\right) = 2\pi i - 3\pi i^2.$$

Since $i^2 = -1$:

$$I = 2\pi i + 3\pi.$$

Final Answer:

$$\boxed{3\pi + 2\pi i}$$

Question 5
Find the Maclaurin series (Taylor series about $z = 0$) of $f(z) = \frac{1}{1+z^2}$. Find the radius of convergence.

Given:

$$f(z) = \frac{1}{1+z^2}$$

To Find:
Maclaurin series expansion about $z = 0$ and its radius of convergence.

Solution:

Step 1: Recall geometric series formula
For $|w| < 1$:

$$\frac{1}{1-w} = \sum_{n=0}^{\infty} w^n .$$

Step 2: Substitute $w = -z^2$

$$f(z) = \frac{1}{1-(-z^2)} = \frac{1}{1-(-z^2)}.$$

Using the formula:

$$f(z) = \sum_{n=0}^{\infty} (-z^2)^n = \sum_{n=0}^{\infty} (-1)^n z^{2n}.$$

Step 3: Write explicit series

$$f(z) = 1 - z^2 + z^4 - z^6 + z^8 - \cdots$$

Step 4: Radius of convergence
The geometric series converges when $|w| < 1$, i.e., $|-z^2| < 1 \implies |z^2| < 1 \implies |z| < 1$.

Alternatively, note singularities occur when $1 + z^2 = 0 \implies z = \pm i$.
The distance from expansion point $z = 0$ to nearest singularity is $|i - 0| = 1$.
Thus radius of convergence $R = 1$.

Final Answer:

Maclaurin series:

$$\boxed{\frac{1}{1+z^2} = \sum_{n=0}^{\infty} (-1)^n z^{2n}, |z| < 1}$$

Radius of convergence:

$$\boxed{R = 1}$$

Question 6
Find the Maclaurin series (Taylor series about $z = 0$) of $f(z) = \frac{1}{z+3i}$. Find the radius of convergence.

Given:

$$f(z) = \frac{1}{z + 3i}$$

To Find:
Maclaurin series about $z = 0$ and its radius of convergence.

Solution:

Step 1: Rewrite in form suitable for geometric series

$$f(z) = \frac{1}{3i + z} = \frac{1}{3i} \cdot \frac{1}{1 + \frac{z}{3i}}.$$

Step 2: Apply geometric series
For $|\, w \,| < 1$, $\frac{1}{1-w} = \sum_{n=0}^{\infty} w^n$.
Let $w = -\frac{z}{3i}$. Then:

$$\frac{1}{1 + \frac{z}{3i}} = \frac{1}{1 - (-\frac{z}{3i})} = \sum_{n=0}^{\infty} \left(-\frac{z}{3i} \right)^n.$$

Step 3: Simplify

$$f(z) = \frac{1}{3i} \sum_{n=0}^{\infty} \left(-\frac{z}{3i} \right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{(3i)^{n+1}} z^n.$$

Note: $(3i)^{n+1} = 3^{n+1} i^{n+1}$, and $i^{n+1} = i^n \cdot i$.

We can also separate real/imaginary patterns, but the compact form is acceptable.

Step 4: Write first few terms

$$f(z) = \frac{1}{3i} - \frac{z}{(3i)^2} + \frac{z^2}{(3i)^3} - \frac{z^3}{(3i)^4} + \cdots$$

Simplify powers of i:

- $1/(3i) = -i/3$
- $1/(3i)^2 = -1/9$
- $1/(3i)^3 = i/27$
- $1/(3i)^4 = 1/81$
  Thus:

$$f(z) = -\frac{i}{3} - \frac{1}{9} z + \frac{i}{27} z^2 + \frac{1}{81} z^3 - \cdots$$

Step 5: Radius of convergence
Singularity at $z = -3i$.
Distance from expansion point $z = 0$ to $-3i$ is $|0 - (-3i)| = |3i| = 3$.
Thus radius of convergence $R = 3$.

Final Answer:

Maclaurin series:

$$\boxed{\frac{1}{z+3i} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(3i)^{n+1}} z^n, |z| < 3}$$

or in simplified form:

$$f(z) = -\frac{i}{3} - \frac{z}{9} + \frac{iz^2}{27} + \frac{z^3}{81} - \cdots$$

Radius of convergence:

$$\boxed{R = 3}$$

Question 7
Find the Taylor series of $f(z) = \frac{1}{z}$ with centre 2. Find the radius of convergence.

Given:

$$f(z) = \frac{1}{z}, \text{centre } z_0 = 2$$

To Find:
Taylor series about $z_0 = 2$ and its radius of convergence.

Solution:

Step 1: Rewrite $f(z)$ in terms of $z - 2$

$$\frac{1}{z} = \frac{1}{2+(z-2)} = \frac{1}{2} \cdot \frac{1}{1+\frac{z-2}{2}}.$$

Step 2: Apply geometric series formula
For $|w| < 1, \frac{1}{1-w} = \sum_{n=0}^{\infty} w^n$.
Let $w = -\frac{z-2}{2}$. Then:

$$\frac{1}{1+\frac{z-2}{2}} = \frac{1}{1-\left(-\frac{z-2}{2}\right)} = \sum_{n=0}^{\infty} \left(-\frac{z-2}{2}\right)^n.$$

Step 3: Multiply by $\frac{1}{2}$

$$\frac{1}{z} = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n \frac{(z-2)^n}{2^n} = \sum_{n=0}^{\infty} \frac{(\ \ -1)^n}{2^{n+1}}(z-2)^n.$$

Step 4: Write explicit expansion

$$\frac{1}{z} = \frac{1}{2} - \frac{z-2}{4} + \frac{(z-2)^2}{8} - \frac{(z-2)^3}{16} + \cdots$$

Step 5: Radius of convergence
The geometric series converges when $|\ w\ |< 1$, i.e.,

$$|-\frac{z-2}{2}|< 1 \implies \frac{|\ z-2\ |}{2} < 1 \implies |\ z-2\ |< 2.$$

Also, the function $f(z) = \frac{1}{z}$ has a singularity at $z = 0$.
Distance from centre $z_0 = 2$ to $z = 0$ is 2, so radius of convergence $R = 2$.

Final Answer:

Taylor series about $z = 2$:

$$\boxed{\frac{1}{z} = \sum_{n=0}^{\infty} \frac{(\ \ -1)^n}{2^{n+1}}(z-2)^n, |\ z-2\ |< 2}$$

Radius of convergence:

$$\boxed{R = 2}$$

Question 8
Find the Taylor series of $f(z) = \sin\ z$ with centre $\pi/2$. Find the radius of convergence.

Given:

$$f(z) = \sin\ z, \text{centre } z_0 = \frac{\pi}{2}$$

To Find:
Taylor series about $z_0 = \pi/2$ and its radius of convergence.

Solution:

Step 1: Recall Taylor series formula
For a function analytic at z_0:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n.$$

Step 2: Compute derivatives at $z_0 = \pi/2$

$$f(z) = \sin\, z, f'(z) = \cos\, z, f''(z) = -\sin\, z, f'''(z) = -\cos\, z, f^{(4)}(z) = \sin\, z, \dots$$

At $z = \pi/2$:

- $\sin\,(\pi/2) = 1$
- $\cos\,(\pi/2) = 0$
- $-\sin\,(\pi/2) = -1$
- $-\cos\,(\pi/2) = 0$
- $\sin\,(\pi/2) = 1$ (repeats every 4 derivatives)

Thus:

$$f(\pi/2) = 1, f'(\pi/2) = 0, f''(\pi/2) = -1, f'''(\pi/2) = 0, f^{(4)}(\pi/2) = 1, \dots$$

Step 3: Write series
Only even derivatives (but shifted because of sine's phase) yield non-zero terms. Let's list:

- $n = 0: \frac{1}{0!}(z - \pi/2)^0 = 1$
- $n = 1: \frac{0}{1!}(z - \pi/2)^1 = 0$
- $n = 2: \frac{-1}{2!}(z - \pi/2)^2 = -\frac{(z-\pi/2)^2}{2!}$
- $n = 3: 0$
- $n = 4: \frac{1}{4!}(z - \pi/2)^4$
- $n = 5: 0$
- $n = 6: \frac{-1}{6!}(z - \pi/2)^6$, etc.

So pattern: non-zero terms for even $n = 2k$, with sign $(\quad -1)^k$.

Thus:

$$\sin\, z = \sum_{k=0}^{\infty} \frac{(\quad -1)^k}{(2k)!}\left(z - \frac{\pi}{2}\right)^{2k}.$$

Step 4: Alternative approach via cosine shift
Note $\sin\, z = \cos\, \left(z - \frac{\pi}{2}\right)$.

Maclaurin series for $\cos w$ is $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} w^{2k}$.

Substitute $w = z - \frac{\pi}{2}$:

$$\sin z = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \left(z - \frac{\pi}{2}\right)^{2k}.$$

Step 5: Radius of convergence

$\sin z$ is entire (analytic everywhere), so the Taylor series converges for all finite z.
Thus radius of convergence $R = \infty$.

Final Answer:

Taylor series about $z = \pi/2$:

$$\boxed{\sin z = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \left(z - \frac{\pi}{2}\right)^{2k}, |z - \pi/2| < \infty}$$

Radius of convergence:

$$\boxed{R = \infty}$$