Subject Name & Code:

MATHEMATICS II- BE02R00011

Assignment – 14

Question 1

Define: Singular point, Isolated singular point, Residue.

Singular point (or singularity)

A point $z = a$ is called a **singular point** (or singularity) of a complex function $f(z)$ if f is not analytic at a but is analytic at some points in every neighbourhood of a.

Isolated singular point

A singular point $z = a$ is **isolated** if there exists a neighbourhood $0 <| z - a |< R$ in which f is analytic. That is, a is the only singularity in some punctured disk around a.

Residue

If $z = a$ is an isolated singularity of $f(z)$, then $f(z)$ has a Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - a)^n \text{ for } 0 <| z - a |< R.$$

The coefficient a_{-1} of the term $\frac{1}{z-a}$ is called the **residue** of $f(z)$ at $z = a$, denoted $\text{Res}(f, a)$.
For a simple pole, $\text{Res}(f, a) = \lim_{z \to a}(z - a)f(z)$.
For a pole of order m:

$$\text{Res}(f, a) = \frac{1}{(m - 1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}}[(z - a)^m f(z)].$$

Question 2

Explain types of isolated singular points.

There are three types of isolated singularities:

1. **Removable singularity**

   o All coefficients a_n with $n < 0$ in the Laurent expansion are zero.

   o The limit $\lim_{z \to a} f(z)$ exists and is finite.

   o Example: $f(z) = \frac{\sin z}{z}$ at $z = 0$.

2. **Pole of order m**

- o   In the Laurent expansion, $a_{-m} \neq 0$ and $a_n = 0$ for $n < -m$.

- o $\lim_{z \to a} | f(z) | = \infty$.

- o   Example: $f(z) = \frac{1}{(z-a)^m}$ has a pole of order m at $z = a$.

3. **Essential singularity**

- o   Infinitely many negative-power terms in the Laurent expansion are non-zero.

- o $\lim_{z \to a} f(z)$ does not exist (neither finite nor infinite).

- o   Example: $f(z) = e^{1/z}$ at $z = 0$.

Question 3

Determine and classify all singularities of the following functions:

1. $f(z) = \frac{\sin z}{z}$

Singularity at $z = 0$.
Check limit: $\lim_{z \to 0} \frac{\sin z}{z} = 1$ (finite).
Laurent expansion:

$$\frac{\sin z}{z} = \frac{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$$

No negative powers $\Rightarrow$ **removable singularity**.

2. $f(z) = \frac{z^3}{(z-1)^3(z-3)^2}$

Singularities at $z = 1$ and $z = 3$.

- At $z = 1$: factor $(z-1)^3$ in denominator, no similar factor in numerator $\Rightarrow$ **pole of order 3**.

- At $z = 3$: factor $(z-3)^2$ in denominator, numerator non-zero at $z = 3 \Rightarrow$ **pole of order 2**.

No other singularities in finite plane.

Question 4

Find the Laurent series of $\frac{1}{z^2-3z+2}$ for regions

$$z^2 - 3z + 2 = (z - 1)(z - 2).$$

So $f(z) = \frac{1}{(z-1)(z-2)}$. Use partial fractions:

$$\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}.$$

(i) $0 < | z | < 1$
Both $| z | < 1$ and $| z | < 2$, so expand each term in powers of z:

$$\frac{1}{z-2} = -\frac{1}{2} \cdot \frac{1}{1-z/2} = -\frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n.$$

$$\frac{1}{z-1} = -\frac{1}{1-z} = -\sum_{n=0}^{\infty} z^n.$$

Subtract:

$$f(z) = \left[-\frac{1}{2}\sum_{n=0}^{\infty}\frac{z^n}{2^n} \right] - \left[-\sum_{n=0}^{\infty} z^n \right] = \sum_{n=0}^{\infty}\left(1 - \frac{1}{2^{n+1}}\right)z^n.$$

(ii) $1 <| z |< 2$

For $\frac{1}{z-1}$, factor z: $| z |> 1 \Rightarrow \frac{1}{z-1} = \frac{1}{z} \cdot \frac{1}{1-1/z} = \sum_{m=1}^{\infty}\frac{1}{z^m}.$

For $\frac{1}{z-2}$, factor 2: $| z |< 2 \Rightarrow \frac{1}{z-2} = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n.$

Thus:

$$f(z) = -\frac{1}{2}\sum_{n=0}^{\infty}\frac{z^n}{2^n} - \sum_{m=1}^{\infty}\frac{1}{z^m}.$$

(iii) $| z |> 2$

Both $| z |> 1$ and $| z |> 2$:

$$\frac{1}{z-1} = \frac{1}{z} \cdot \frac{1}{1-1/z} = \sum_{m=1}^{\infty}\frac{1}{z^m},$$

$$\frac{1}{z-2} = \frac{1}{z} \cdot \frac{1}{1-2/z} = \sum_{k=1}^{\infty}\frac{2^{k-1}}{z^k}.$$

Subtract:

$$f(z) = \sum_{k=1}^{\infty}\frac{2^{k-1}-1}{z^k}.$$

Question 5

Find the Laurent series of $\frac{7z-2}{(z+1)z(z-2)}$ for $1 <| z+1 |< 3$.

Let $w = z+1 \Rightarrow z = w-1$. Then:

$$F(w) = \frac{7(w-1)-2}{w(w-1)(w-3)} = \frac{7w-9}{w(w-1)(w-3)}.$$

Partial fractions:

$$\frac{7w - 9}{w(w - 1)(w - 3)} = \frac{A}{w} + \frac{B}{w - 1} + \frac{C}{w - 3}.$$

Solve: $A = 3, B = -1, C = -2$.
So:

$$F(w) = \frac{3}{w} - \frac{1}{w - 1} - \frac{2}{w - 3}.$$

Region: $1 < |w| < 3$.

- For $\frac{3}{w}$: already Laurent term $3w^{-1}$.

- For $-\frac{1}{w-1}$: since $|w| > 1$,

$$-\frac{1}{w - 1} = -\frac{1}{w} \cdot \frac{1}{1 - 1/w} = -\sum_{n=1}^{\infty} \frac{1}{w^{n+1}}.$$

- For $-\frac{2}{w-3}$: since $|w| < 3$,

$$-\frac{2}{w - 3} = \frac{2}{3} \cdot \frac{1}{1 - w/3} = \frac{2}{3} \sum_{m=0}^{\infty} \left(\frac{w}{3}\right)^m.$$

Combine:

$$F(w) = 3w^{-1} - \sum_{n=1}^{\infty} w^{-(n+1)} + \frac{2}{3} \sum_{m=0}^{\infty} \frac{w^m}{3^m}.$$

Replace $w = z + 1$:

$$f(z) = 3(z + 1)^{-1} - \sum_{n=1}^{\infty} (z + 1)^{-(n+1)} + \frac{2}{3} \sum_{m=0}^{\infty} \frac{(z+1)^m}{3^m}.$$

Question 6

Find the residues at singular points of $f(z) = \frac{z^2}{(z-1)^2(z+2)}$.

Singularities: pole of order 2 at $z = 1$, simple pole at $z = -2$.

- At $z = -2$ (simple):

$$\text{Res}(f, -2) = \lim_{z \to -2} (z + 2)f(z) = \frac{(-2)^2}{(-2-1)^2} = \frac{4}{9}.$$

- At $z = 1$ (order 2):

Let $g(z) = (z-1)^2 f(z) = \frac{z^2}{z+2}$. Then:

$$\text{Res}(f, 1) = \frac{1}{(2-1)!} \lim_{z \to 1} g'(z) = g'(1).$$

$$g(z) = \frac{z^2}{z+2} \implies g'(z) = \frac{2z(z+2) - z^2}{(z+2)^2} = \frac{z^2 + 4z}{(z+2)^2}.$$

At $z = 1$: $g'(1) = \frac{1+4}{9} = \frac{5}{9}$.

Thus:

$$\text{Res}(f, -2) = \frac{4}{9}, \text{Res}(f, 1) = \frac{5}{9}.$$

Question 7

Find the residues at singular points of $f(z) = \frac{1}{(z+1)^4}$.

Only singularity: pole of order 4 at $z = -1$.
For pole of order $m = 4$:

$$\text{Res}(f, -1) = \frac{1}{(4-1)!} \lim_{z \to -1} \frac{d^3}{dz^3} [(z+1)^4 f(z)].$$

But $(z+1)^4 f(z) = 1$, whose third derivative is 0.
Hence $\text{Res}(f, -1) = 0$.

Question 8

Evaluate $\int_C \frac{dz}{z^3(z+4)}$ using Cauchy residue theorem, where

(a) $C: |z| = 2$

Singularities inside C: $z = 0$ (order 3), $z = -4$ (simple) is outside ($|-4| = 4 > 2$).
Only $z = 0$ contributes.

For $g(z) = \frac{1}{z+4}$, analytic near 0:

$$f(z) = \frac{g(z)}{z^3} \implies \text{Res}(f, 0) = \frac{g''(0)}{2!}.$$

$$g(z) = \frac{1}{z+4} \implies g'(z) = -\frac{1}{(z+4)^2}, g''(z) = \frac{2}{(z+4)^3}.$$

$$g''(0) = \frac{2}{64} = \frac{1}{32}.$$

Thus $\text{Res}(f, 0) = \frac{1/32}{2} = \frac{1}{64}$.

By residue theorem:

$$\int_{|z|=2} f(z)\, dz = 2\pi i \cdot \frac{1}{64} = \frac{\pi i}{32}.$$

(b) $C:\ |\ z + 2\ | = 3$

Singularities: $z = 0$ and $z = -4$.
Circle centre -2, radius 3:

- Distance from centre to 0: $|\ 0 + 2\ | = 2 < 3 \Rightarrow$ inside.

- Distance from centre to -4: $|\ -4 + 2\ | = 2 < 3 \Rightarrow$ inside.

Both singularities are inside.

Residue at $z = -4$ (simple):

$$\mathrm{Res}(f, -4) = \lim_{z \to -4} (z + 4) f(z) = \frac{1}{(\ -4)^3} = -\frac{1}{64}.$$

Residue at $z = 0$ computed earlier: $\frac{1}{64}$.

Sum of residues: $\frac{1}{64} - \frac{1}{64} = 0$.

Thus:

$$\int_{|z+2|=3} f(z)\, dz = 2\pi i \cdot 0 = 0.$$

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