UNIT – 4

Geometric Transformations

Contents

Important Repeated Questions:

1. **Differentiate between/Compare Constructive Solid Geometry (CSG) and B-Representation (B-Rep).** (W25 - Q2b, 04 marks) (S23 - Q4a, 03 marks) (S25 - Q2a, 03 marks)
2. **Explain Homogeneous Coordinate system and its importance/advantages.** (W25 - Q3a, 03 marks) (W22 - Q3a OR, 03 marks) (S25 - Q3a OR, 03 marks)
3. **Write/Explain transformation matrices (Rotation, Reflection, Shearing, etc.) in 2D/3D using homogeneous coordinates.** (W25 - Q3a OR, 03 marks) (W22 - Q4b, 04 marks) (S24 - Q5a, 03 marks)
4. **Differentiate between/Compare Scaling and Shearing transformation.** (W25 - Q3b, 04 marks)
5. **Find the new coordinates of a shape (triangle/rectangle) after applying a series of transformations (translation, rotation, scaling, reflection).** (S24 - Q3c OR, N/A marks) (W22 - Q3c OR, 07 marks) (W23 - Q3c, N/A marks) (W24 - Q3c OR, N/A marks)
6. **Obtain the mirror reflection of a triangle/polygon about a given line.** (W25 - Q3c, 07 marks) (S25 - Q3c, 07 marks) (S22 - Q4c, N/A marks)
7. **Prove that two successive rotations/translations/scaling are commutative.** (W25 - Q3c OR, 07 marks)
8. **Prove that differential scaling and rotation are not commutative, but uniform scaling and rotation are commutative.** (S23 - Q4c, 07 marks)
9. **What do you understand by geometry and topology in solid modelling?** (S23 - Q1c, 07 marks) (W23 - Q5a OR, 03 marks)
10. **Explain Boolean operations for Constructive Solid Geometry (CSG).** (S23 - Q4b, 04 marks) (S25 - Q2b, 04 marks)
11. **Compare Wireframe, Surface and Solid modeling techniques.** (W22 - Q3b OR, 04 marks)
12. **Explain Feature based modeling.** (S24 - Q5c, N/A marks) (W24 - Q4c, 07 marks)
13. **List methods of geometric modeling. Explain Wire frame modeling.** (S23 - Q3b OR, 04 marks)

Legends: W- Winter, S- Summer, Q- Question and 03/04/07- Marks of Question

4.1 Geometric Transformations

- All changes performed on the graphic image are done by changing the database of the original picture. These changes are called as transformations.
- Transformations allow the user to uniformly change the entire picture. An object created by the user is stored in the form of a database. If the database, which represents the object, is changed, the object also would change. This method is used to alter the orientation, scale, position of the drawing.
- In general, geometric transformations can be defined as the change done to the database by performing certain mathematical operations on it, so as to produce the desired change in the image.

4.1.1 Translation

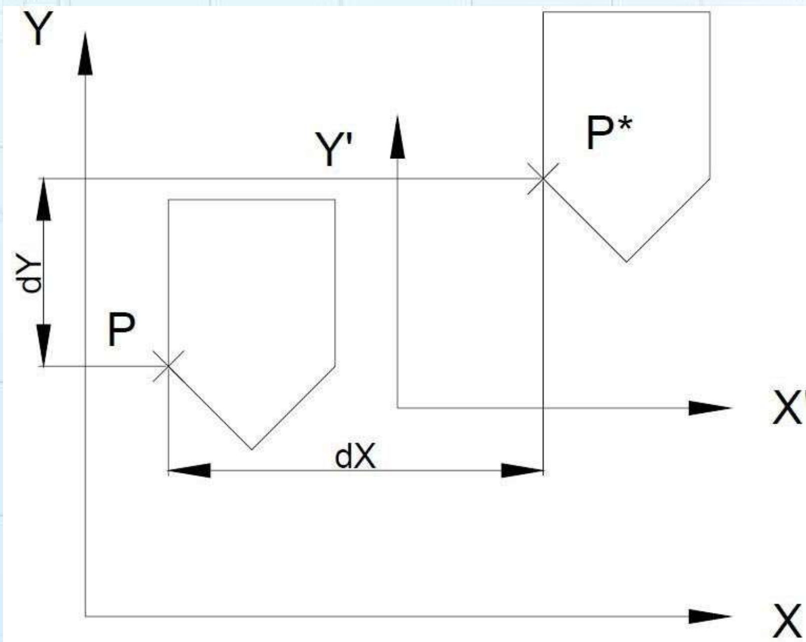


Fig.4.1 – Translation

- When every entity of a geometric model remains parallel to its initial position, the transformation is called as translation.
- Translating a model therefore implies that every point on it moves by an equal given distance in a given direction. A translation involves moving of an element from one location to another.

$$P^* = [x^* + y^*]$$

$$x^* = x + dx$$

$$y^* = y + dy$$

$$[P^*] = \begin{bmatrix} x^* \\ y^* \end{bmatrix}$$

$$= \begin{bmatrix} x + dx \\ y + dy \end{bmatrix}$$

$$= \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} dx \\ dy \end{bmatrix}$$

4.1.2 Scaling

- Scaling transformation alters the size of an object. Scaling can be uniform (i.e. equal in both X and Y directions) or non-uniform (i.e. different in X and Y directions).
- To achieve scaling, the original coordinates would be multiplied by scaling factor.

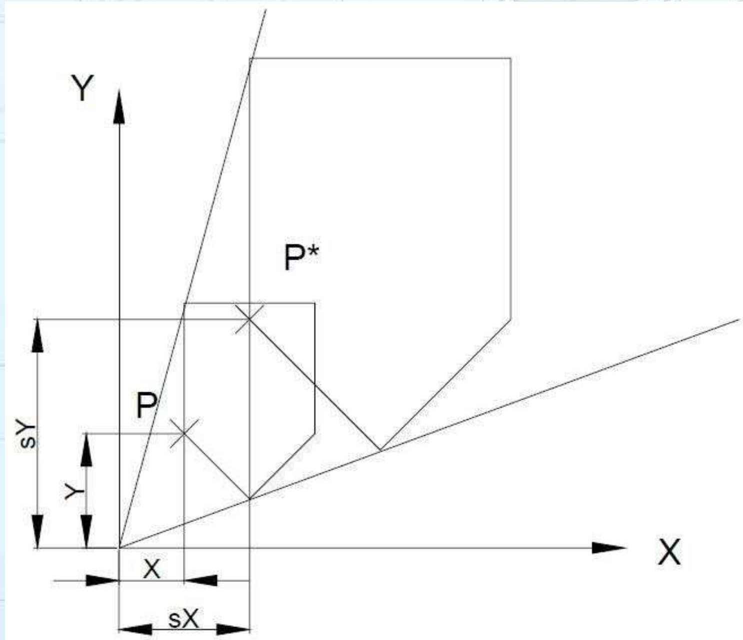


Fig.4.2 – Scaling

$$P^* = [x^*, y^*]$$

$$= [S_x \times x, S_y \times y]$$

- This equation can also be represented in a matrix form as follow:

$$[P^*] = \begin{bmatrix} S_x & 0 \\ 0 & S_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= [T_S][P]$$

Where scalling transformation, $[T_S] = \begin{bmatrix} S_x & 0 \\ 0 & S_y \end{bmatrix}$

4.1.3 Reflection or Mirror

- Reflection is the process of obtaining a mirror of the original shape.
- This is an important transformation and is used quite often as many engineered products are symmetrical.
- The following transformation matrices as shown in *Fig.4.3* when multiplied to the original point produce a Reflection or Mirror.
- Thus, in general,

$$P' \square P \square M$$

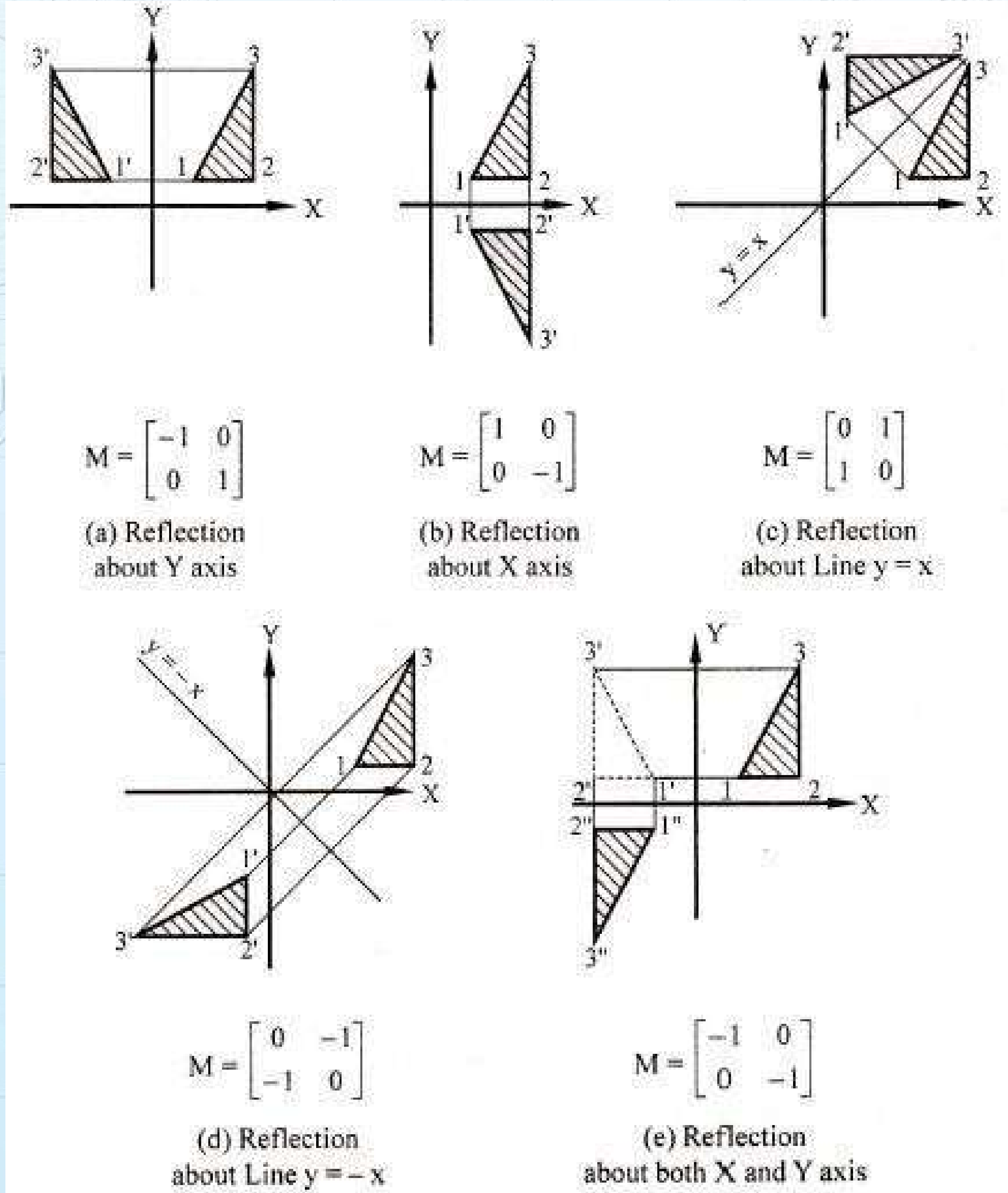


Fig.4.3 – Reflection

4.1.4 Rotation

- Rotation is an important transformation and allows the user to view the objects from different angles.
- It can also be used to create entities arranged in circular arrays, by creating the entity once and then rotating / copying it to the desired positions on the circumference. This would allow the user to create an array of objects.

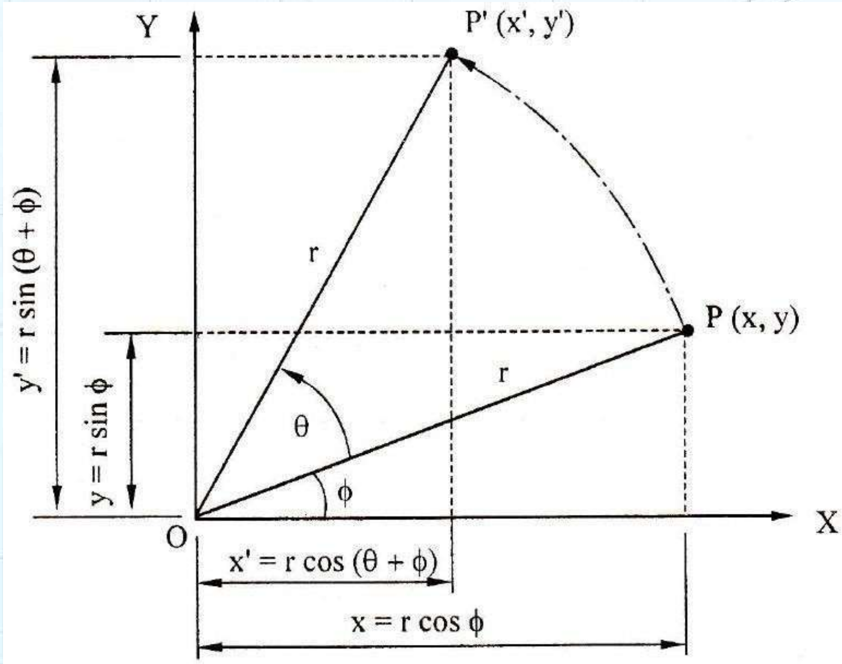


Fig.4.4 – Rotation

- Let point P (x, y) be rotated by an angle θ about the origin O.

$$P = [x \quad y] = [r \cos \emptyset \quad r \sin \emptyset]$$

- Let the rotated point be represented as :

$$P' = [x' \quad y'] = [r \cos(\theta + \emptyset) \quad r \sin(\theta + \emptyset)]$$

$$P' = [r\,(\cos\theta \cos\emptyset - \sin\theta \sin\emptyset) \quad r(\sin\theta\, cos\emptyset + \cos\theta \sin\emptyset)]$$

$$P' = [(x \cos\theta - y \sin\theta) \quad (x \sin\theta + y \cos\theta)]$$

- This can be expressed as:

$$P' = [x \quad y] \begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix}$$

4.2 Homogeneous Representation

- Many a times it becomes necessary to combine the above mentioned individual transformations in order to achieve the required results. In such cases the combined transformation matrix can be obtained by multiplying the respective transformation matrices.

- However, care should to be taken that the order of the matrix multiplication be done in the same way as that of the transformations as follows.

$$[P^*] = [T_n][T_{n-1}][T_{n-2}] \ldots\ldots [T_3][T_2][T_1] \qquad \text{Eq. (4.1)}$$

- In *Eq. (4.1)* all the transformation matrices should be multiplicative type.
- The following form should be used to convert the translation into a multiplication form.

$$P^* = [x^* \quad y^* \quad 1]$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ dX & dY & 1 \end{bmatrix} [x \quad y \quad 1]$$

- Hence the translation matrix in multiplication form can be given as

$$[M_T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ dX & dY & 1 \end{bmatrix}$$

- This is termed as homogeneous representation. In homogeneous representation, an n-dimensional space is mapped into (n+1) dimensional space. Thus a 2 dimensional point [x y] is represented by 3 dimensions as [x y 1]. Homogeneous representation can be experienced in the following situations.

4.2.1 Rotation about an arbitrary Point

- The transformation given earlier for rotation is about the origin of the axes system. It may sometimes be necessary to get the rotation about any arbitrary base point as shown in *Fig.4.5*. To derive the necessary transformation matrix, the following complex procedure comprising the following three points would be required.

  (i) Translate the point A to O, the origin of the axes system.

  (ii) Rotate the object by the given angle.

  (iii) Translate the point back to its original position.

- The transformation matrices for the above operations in the given sequence are the following.

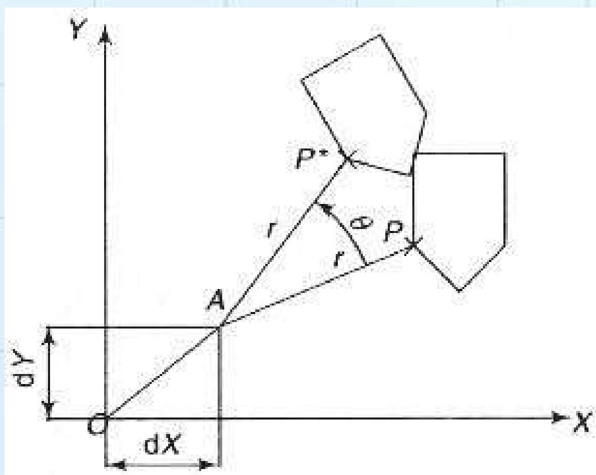


Fig.4.5 – Rotation about an arbitrary Point

$$[T_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ dX & dY & 1 \end{bmatrix}$$

$$[T_2] = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[T_3] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -dX & -dY & 1 \end{bmatrix}$$

- The required transformation matrix is given by [T]

  = [T_3] [T_2] [T_1]

$$[T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -dX & -dY & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ dX & dY & 1 \end{bmatrix}$$

4.2.2 Reflection about an arbitrary line

- Similar to the above, there are times when the reflection is to be taken about an arbitrary line as shown in *Fig.4.6*.

1. Translate the mirror line along Y axis such that line passes through the origin, O.
2. Rotate the mirror line such that it coincides with the X axis.
3. Mirror the object through the X axis.
4. Rotate the mirror line back to the original angle with X axis.
5. Translate the mirror line along the Y axis back to the original position. Following are

the transformation matrices for the above operations in the given sequence.

Fig.4.6 – Reflection Transformation about on arbitrary Line

$$[T_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & C & 1 \end{bmatrix}$$

$$[T_2] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[T_3] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[T_4] = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[T_5] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -C & 1 \end{bmatrix}$$

- The required transformation matrix is given by

$$[T] = [T_5]\,[T_4]\,[T_3]\,[T_2]\,[T_1]$$

$$[T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -C & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & C & 1 \end{bmatrix}$$

4.3 Shearing

- A transformation that distorts the shape of an object such that the transformed shape appears as if the object were composed of internal layers that had been caused to slide over each other is called a shear.

4.3.1 2D Shearing

- Two common shearing transformations are those that shift coordinate x values and those that shift y values.
- An x-direction shear relative to the x axis is produced with the transformation matrix

$$\begin{bmatrix} 1 & Sh_x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{Eq. (4.2)}$$

- which transforms coordinate positions as

$$x' = x + Sh_x \cdot y \qquad y' = y \qquad \text{Eq. (4.3)}$$

- Any real number can be assigned to the shear parameter sh_x. A coordinate position (x, y) is then shifted horizontally by an amount proportional to its distance (y value) from the x axis (y = 0).
- Setting sh_x to 2, for example, changes the square in *Fig.4.7* into a parallelogram. Negative values for sh_x shift coordinate positions to the left.

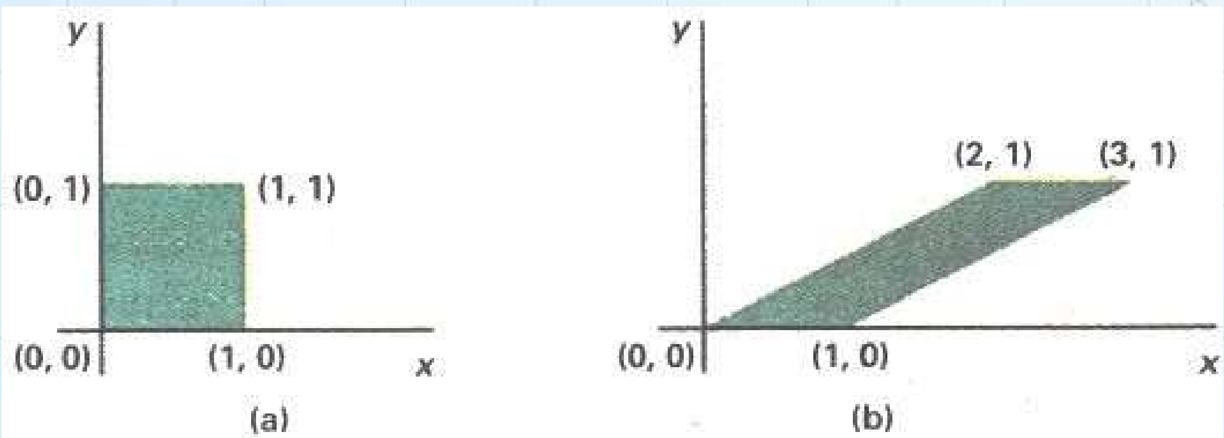


Fig.4.7 – A unit square (a) is converted to a parallelogram (b) using the x direction shear matrix Eq. (4.2) with sh_x = 2.

- We can generate x-direction shears relative to other reference lines with

$$\begin{bmatrix} 1 & Sh_x & -Sh_x \cdot y_{ref} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{Eq. (4.4)}$$

- with coordinate positions transformed as

$$x' = x + Sh_x\,(y - y_{ref}) \qquad y' = y \qquad \text{Eq. (4.5)}$$

- An example of this shearing transformation is given in *Fig.4.8* for a shear parameter value of 1/2 relative to the line $y_{ref} = -1$.

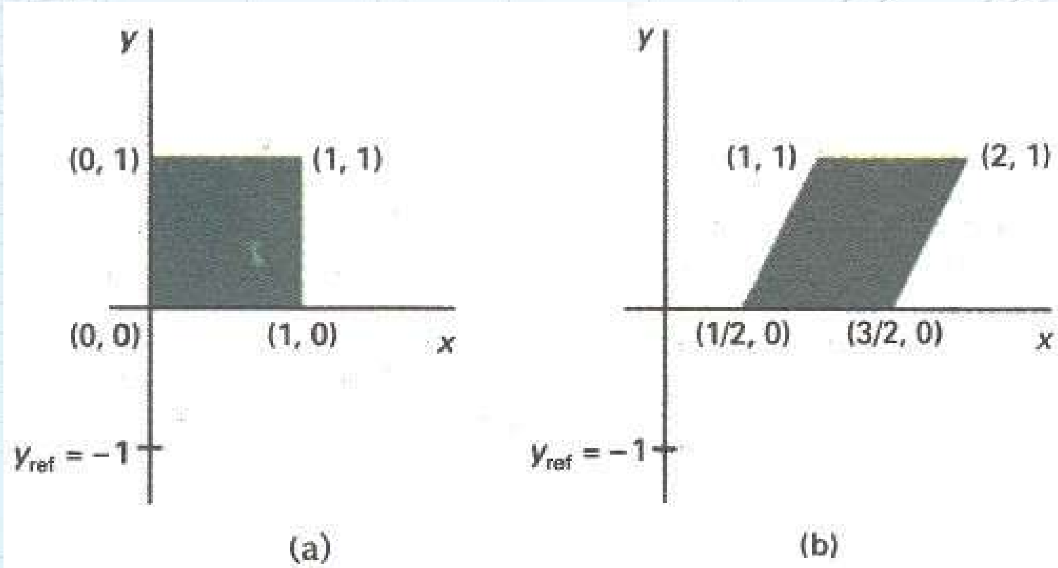


Fig.4.8 – A unit square (a) is transformed to a shifted parallelogram (b) with $sh_x = 1/2$ and $y_{ref} = -1$ in the shear matrix Eq. (4.4).

- A y-direction shear relative to line $x = x_{ref}$ is generated with transformation matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ Sh_y & 1 & -Sh_y . x_{ref} \\ 0 & 0 & 1 \end{bmatrix} \qquad Eq.\ (4.6)$$

- which generates transformed coordinate positions

$$x' = x \qquad y' = y + Sh_y\,(x - x_{ref}) \qquad Eq.\ (4.7)$$

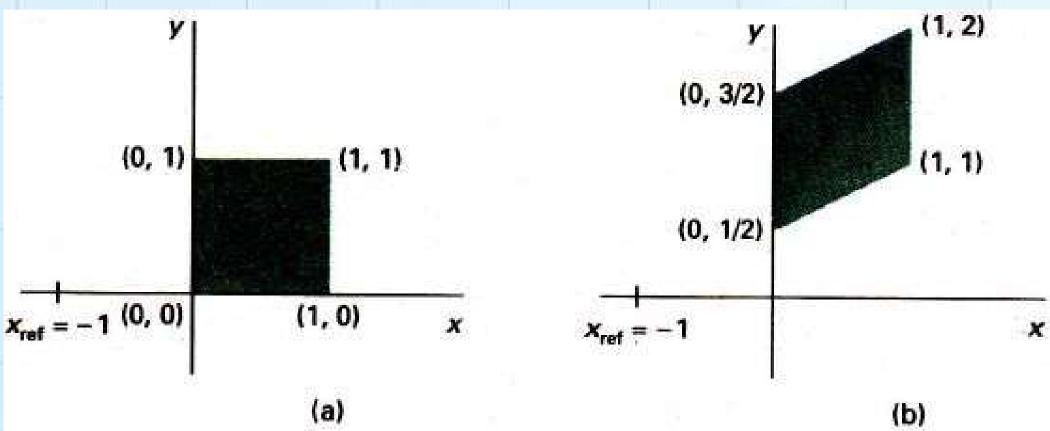


Fig.4.9 – A unit square (a) is turned into a shifted parallelogram (b) with parameter values $sh_y = 1/2$ and $x_{ref} = -1$ in the y direction using shearing transformation Eq. (4.6).

- This transformation shifts a coordinate position vertically by an amount proportional to its distance from the reference line $x = x_{ref}$. *Fig.4.9* illustrates the conversion of a square into a parallelogram with $sh_y = 1/2$ and $x_{ref} = -1$.
- Shearing operations can be expressed as sequences of basic transformations. The x-direction shear matrix *Eq. (4.2)*, for example, can be written as a composite transformation involving a series of rotation and scaling matrices that would scale the unit square of *Fig.4.9* along its diagonal, while maintaining the original lengths and orientations of edges parallel to the x axis.
- Shifts in the positions of objects relative to shearing reference lines are equivalent to translations.

4.3.2 3D Shearing

- In two dimensions, transformations relative to the x or y axes to produce distortions in the shapes of objects. In three dimensions, we can also generate shears relative to the z axis.

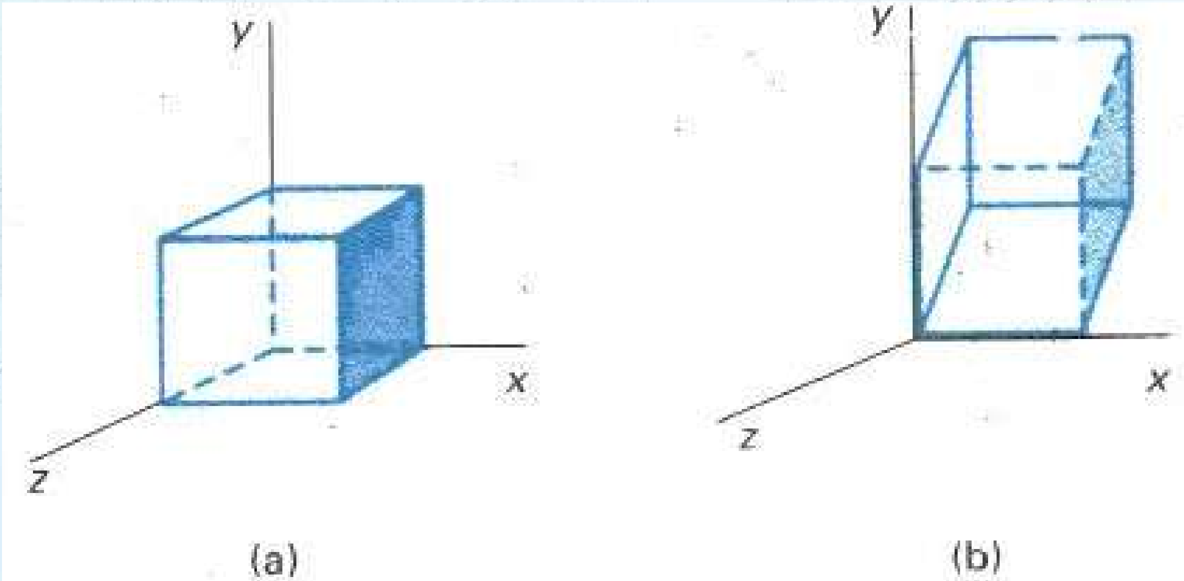


Fig.4.10

- As an example of three-dimensional shearing, the following transformation produces a z-axis shear:

$$[SH_Z] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a & b & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Parameters a and b can be assigned any real values. The effect of this transformation matrix is to alter x- and y-coordinate values by an amount that is proportional to the z value, while leaving the z coordinate unchanged.
- Boundaries of planes that are perpendicular to the z axis are thus shifted by an amount proportional to z. An example of the effect of this shearing matrix on a unit cube is shown in *Fig.4.10*, for shearing values a=b=1. Shearing matrices for the x axis and y axis are defined similarly.

4.4 Projections of Geometric Models

- Databases of geometric models can only be viewed and examined if they can be displayed in various views on a display device or screen. Viewing a three dimensional model is a rather complex process due to the fact that display devices can only display graphics on two-dimensional screens.
- This mismatch between three-dimensional models and two-dimensional screens can be resolved by utilizing projections that transform three-dimensional models onto a two-dimensional projection plane. Various views of a model can be generated using various projection planes.
- To define a projection, a center of projection and a projection plane must be defined as shown in *Fig.4.11*. To obtain the projection of an entity (a line connecting points P_1 and P_2 in the figure), projection rays (called projectors) are constructed by connecting the center of projection with each point of the entity.

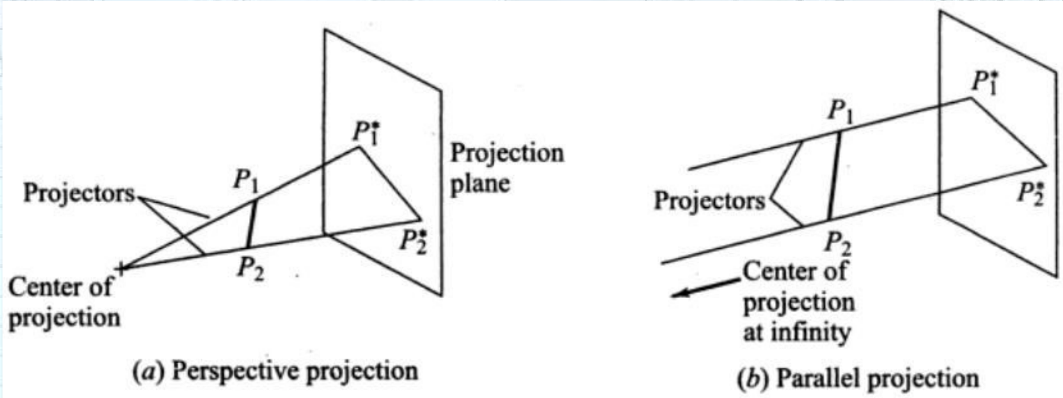


Fig.4.11 – Projection Definition

- The intersections of these projectors with the projection plane define the projected points which are connected to produce the projected entity. There are two different types of projections based on the location of the center of projection relative to the projection plane.
- If the center is at a finite distance from the plane, perspective projection results and all the projectors meet at the center. If, on the other hand, the center is at an infinite distance, all the projectors become parallel (meet at infinity) and parallel projection results. Perspective projection is usually a part of perspective, or projective, geometry.
- Such geometry does not preserve parallelism, that is, no two lines are parallel. Parallel projection is a part of affine geometry which is identical to Euclidean geometry. In affine geometry, parallelism is an important concept and therefore is preserved.

4.4.1 Perspective projection

- Perspective projection creates an artistic effect that adds some realism to perspective views. As can be seen from *Fig.4.11*(a), the size of an entity is inversely proportional to its distance from the center of projection; that is, the closer the entity to the center, the larger its size is.
- Perspective views are not popular among engineers and draftsmen because actual dimensions and angles of objects and therefore shapes, cannot be preserved, which implies that measurements cannot be taken from perspective views directly. In addition, perspective projection does not preserve parallelism.
- In order to define perspective projection, a centre of projection and a projection plane are required. The centre of projection is placed along the z-axis and the image is projected onto z = 0 or xy plane.
- The transformation of perspective projection is given by: P* = Po P

where

$$P_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a & b & 0 & 0 \\ 0 & 0 & -1/d & 1 \end{bmatrix}$$

P is the given point.

P* is the reflected point.

d is the distance between the centre of projection to the projection plane.

4.4.2 Orthographic Projection

- Unlike perspective projection, parallel projection preserves actual dimensions and shapes of objects. It also preserves parallelism. Angles are preserved only on faces of the object which are parallel to the projection plane.
- There are two types of parallel projections based on the relation between the direction of projection and the projection plane. If this direction is normal to the projection plane, orthographic projection and views result.
- The orthographic projection system is to include a total of six projecting planes in any direction required for complete description.
- A typical example is shown in *Fig.4.12* where the object is enclosed in a box such that there are 6 mutually perpendicular projecting planes on which all possible 6 views of the object can be projected.
- This would help in obtaining all the details of the object as shown in *Fig.4.13*. The visible lines are shown with the help of continuous lines while those that are not visible are shown by means of broken lines.

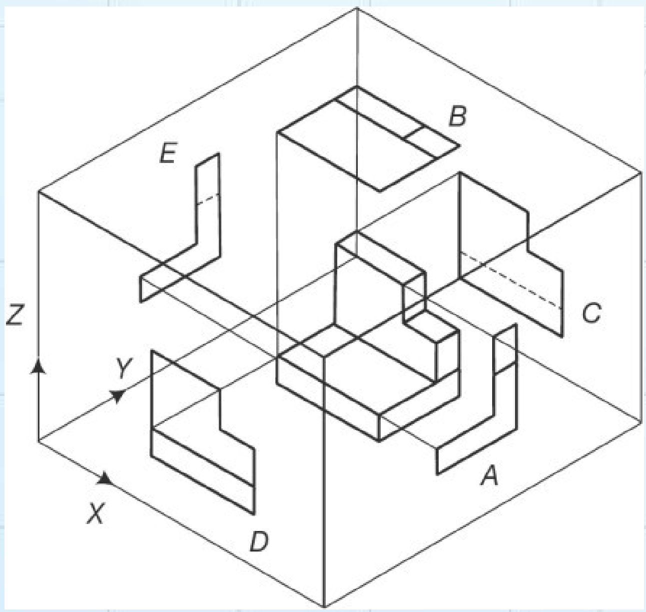


Fig.4.12 – An object Enclosed in a Cube for Obtaining Various Parallel Projections

- Obtaining the orthographic projections is relatively straight forward because of the parallel projections involved. The top view can be obtained by setting z = 0. The transformation matrix will then be

$$[M_{TOP}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- For obtaining the front view, y = 0 and then the resulting coordinates (x, z) will be rotated by 90^0 such that the Z axis coincides with the Y axis. The transformation matrix will then be

$$[M_{FRONT}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

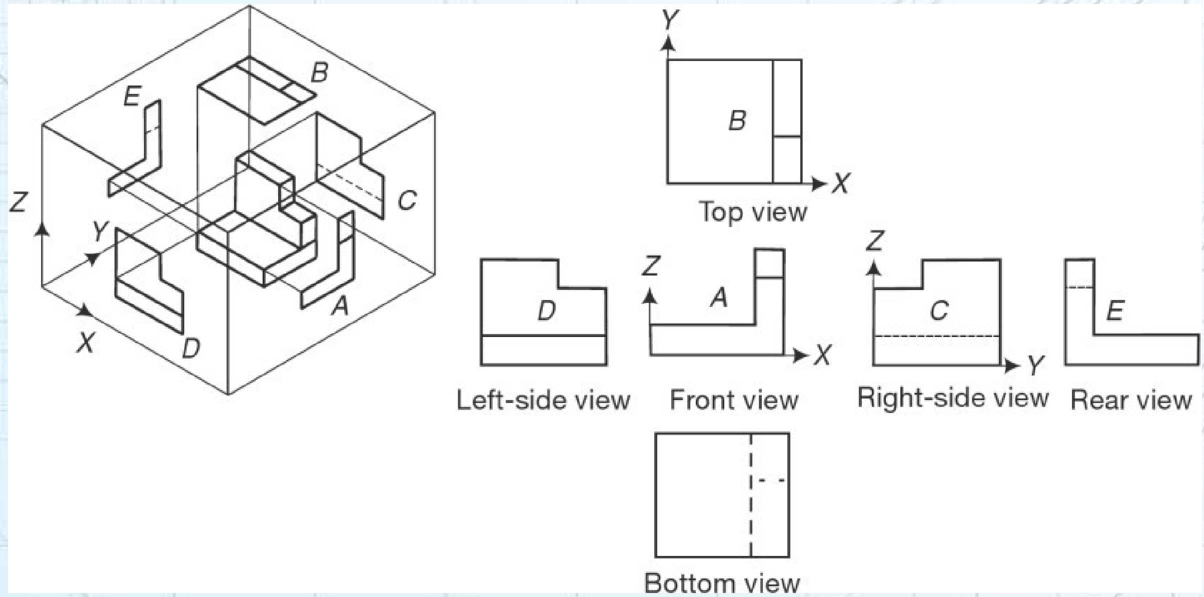


Fig.4.13 – Orthographic Projection of an Object.

- Similarly for obtaining the right side view, x = 0 and then the coordinate system is rotated such that Y axis coincides with the X axis and the Z axis coincides with the Y axis. The transformation matrix will then be

$$[M_{RIGHT}] = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

4.5 Window to View-port transformation

- A world-coordinate area selected for display is called a window. An area on a display device to which a window is mapped is called a viewport. The window defines what is to be viewed; the viewport defines where it is to be displayed.
- Often, windows and viewports are rectangles in standard position, with the rectangle edges parallel to the coordinate axes. Other window or viewport geometries, such as general polygon shapes and circles, are used in some applications, but these shapes take longer to process.

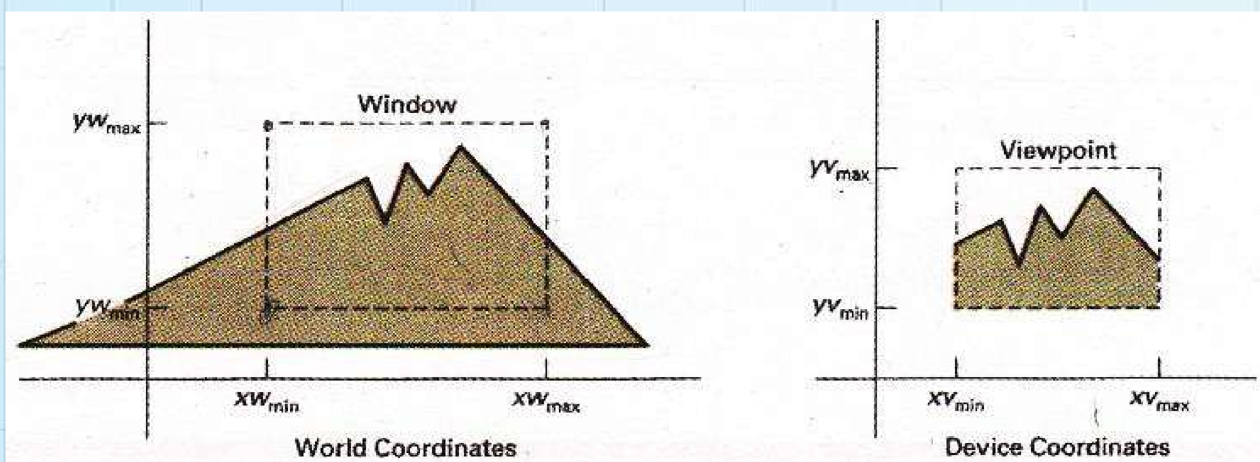


Fig.4.14 – A viewing transformation using standard rectangles for the window and viewport

- In general, the mapping of a part of a world-coordinate scene to device coordinates is referred to as a viewing transformation. Sometimes the two-dimensional viewing transformation is simply referred to as the window-to-viewport transformation or the windowing transformation.
- *Fig.4.14* illustrates the mapping of a picture section that falls within a rectangular window onto a designated rectangular viewport.
- *Fig.4.15* illustrates the window-to-viewport mapping. A point at position (xw, yw) in the window is mapped into position (xv,yv) in the associated viewport. To maintain the same relative placement in the viewport as in the window, we require that

$$\frac{xv - xv_{min}}{xv_{max} - xv_{min}} = \frac{xw - xw_{min}}{xw_{max} - xw_{min}}$$

$$\frac{yv - yv_{min}}{yv_{max} - yv_{min}} = \frac{yw - yw_{min}}{yw_{max} - yw_{min}} \qquad \text{Eq. (4.8)}$$

- Solving these expressions for the viewport position (xv, yv), we have

$$xv = xv_{min} + (xw - xw_{min})sx \qquad \text{Eq. (4.9)}$$

$$yv = yv_{min} + (yw - yw_{min})sy$$

- Where the scaling factors are

$$sx = \frac{xv_{max} - xv_{min}}{xw_{max} - xw_{min}} \qquad \text{Eq. (4.10)}$$

$$sy = \frac{yv_{max} - yv_{min}}{yw_{max} - yw_{min}}$$

- *Eq. (4.9)* can also be derived with a set of transformations that converts the window area into the viewport area. This conversion is performed with the following sequence of transformations:

1. Perform a scaling transformation using a fixed-point position of (xw_{min}, yw_{min}) that scales the window area to the size of the viewport.
2. Translate the scaled window area to the position of the viewport.

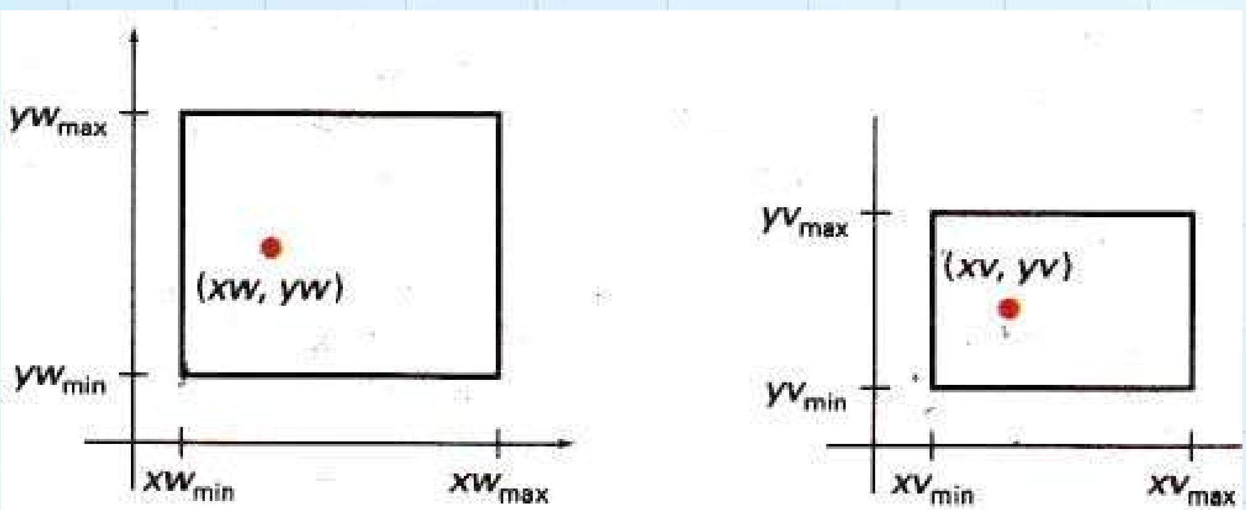


Fig.4.15 – A point at position (x_w, y_w) in a designated window is mapped to viewport coordinates (x_v,y_v) so that relative positions in the two areas are the same.

- Relative proportions of objects are maintained if the scaling factors are the same (sx = sy). Otherwise, world objects will be stretched or contracted in either the x or y direction when displayed on the output device.
- Character strings can be handled in two ways when they are mapped to a viewport. The simplest mapping maintains a constant character size, even though the viewport area may be enlarged or reduced relative to the window.
- This method would be employed when text is formed with standard character fonts that cannot be changed. In systems that allow for changes in character size, string definitions can be windowed the same as other primitives. For characters formed with line segments, the mapping to the viewport can be carried out as a sequence of line transformations.
- From normalized coordinates, object descriptions are mapped to the various display devices. Any number of output devices can be open in a particular application, and another window-to-viewport transformation can be performed for each open output device.
- This mapping, called the workstation transformation, is accomplished by selecting a window area in normalized space and a viewport area in the coordinates of the display device.
- With the workstation transformation, we gain some additional control over the positioning of a scene on individual output devices. As illustrated in *Fig.4.16*, we can use workstation transformations to partition a view so that different parts of normalized space can be displayed on different output devices.

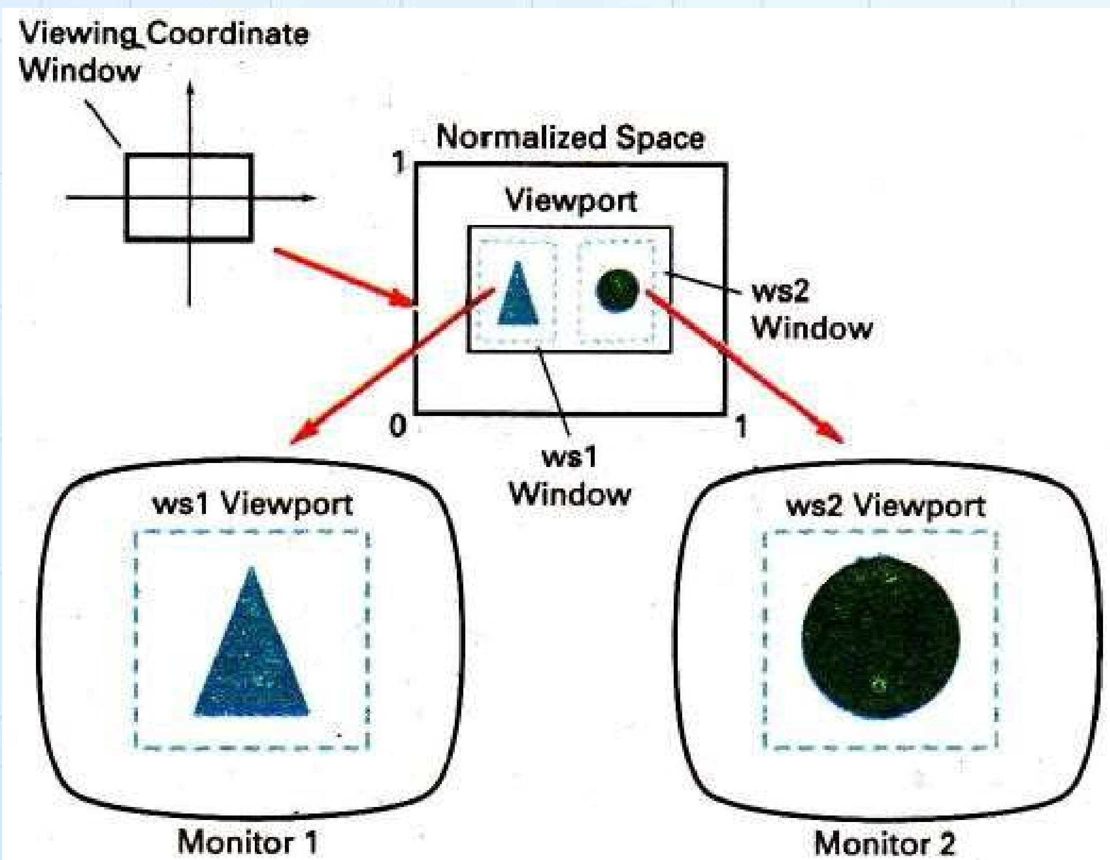


Fig.4.16 – Mapping selected parts of a scene in normalized coordinates to different video monitors with workstation transformations.

Ex. 4.1 ***[GTU; Nov-2011; 7 Marks]***

A triangle ABC with vertices A (32, 22), B (88, 20) and C (32, 82) is to be scaled by a factor of 0.6 about a point X (50, 42). Determine the co-ordinates of the vertices for the scaled triangle.

Solution: Data given:

$S_x = S_y = 0.6$

Scaling about a point X (50, 42)

So, first translate the object to origin then scale the object and then translate back the scaled object to the given point X (50, 42).

$$\Delta A'B'C' = \Delta ABC.\,T.\,S.\,T^{-1}$$

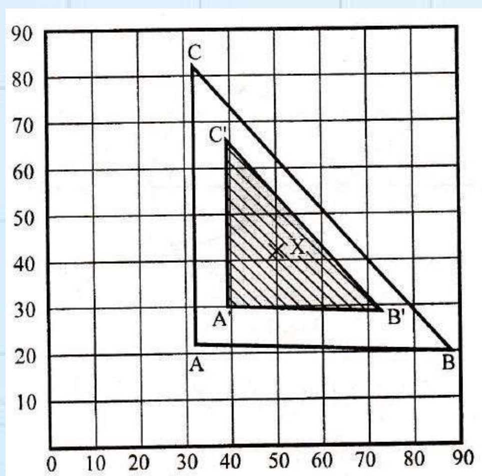
$$\Delta A'B'C' = \begin{bmatrix} 32 & 22 & 1 \\ 88 & 20 & 1 \\ 32 & 82 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -50 & -42 & 1 \end{bmatrix} \begin{bmatrix} 0.6 & 0 & 0 \\ 0 & 0.6 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 50 & 42 & 1 \end{bmatrix}$$

$$\Delta A'B'C' = \begin{bmatrix} 32 & 22 & 1 \\ 88 & 20 & 1 \\ 32 & 82 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -50 & -42 & 1 \end{bmatrix} \begin{bmatrix} 0.6 & 0 & 0 \\ 0 & 0.6 & 0 \\ 50 & 42 & 1 \end{bmatrix}$$

$$\Delta A'B'C' = \begin{bmatrix} 32 & 22 & 1 \\ 88 & 20 & 1 \\ 32 & 82 & 1 \end{bmatrix} \begin{bmatrix} 0.6 & 0 & 0 \\ 0 & 0.6 & 0 \\ 20 & 16.8 & 1 \end{bmatrix}$$

$$\Delta A'B'C' = \begin{bmatrix} 39.2 & 30 & 1 \\ 72.8 & 28.8 & 1 \\ 39.2 & 66 & 1 \end{bmatrix}$$

Answer: New vertices are A’ (39.2, 30), B’ (72.8, 28.8) and C’ (39.2, 66).



Ex. 4.2 ***[GTU; Nov-2011; 7 Marks]***

Using transformation matrix, determine the new co-ordinates of triangle ABC with vertices A (0, 0), B (3, 2) and C (2, 3) after it is rotated by 45° clockwise about origin.

Solution: Data given:

Rotation about origin 45° clockwise Hence, θ

$= -45°$

$$\Delta A'B'C' = \Delta ABC.\,R$$

$$\Delta A'B'C' = \begin{bmatrix} 0 & 0 & 1 \\ 3 & 2 & 1 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} Cos(-45°) & Sin(-45°) & 0 \\ -Sin(-45°) & Cos(-45°) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Delta A'B'C' = \begin{bmatrix} 0 & 0 & 1 \\ 3 & 2 & 1 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 0.707 & -0.707 & 0 \\ 0.707 & 0.707 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Delta A'B'C' = \begin{bmatrix} 0 & 0 & 1 \\ 3.536 & -0.707 & 1 \\ 3.536 & 0.707 & 1 \end{bmatrix}$$

Answer: New vertices are A’ (0, 0), B’ (3.536, –0.707) and C’ (3.536, 0.707).

Ex. 4.3 ***[GTU; Nov-2011; 7 Marks]***

A rectangle ABCD has vertices A (1, 1), B (2, 1), C (2, 3) and D (1, 3). It has to be rotated by 30° counter clockwise about a point P (3, 2). Determine the new co-ordinates of the vertices for the rectangle.

Solution: Data given:

Rotation about a point 45° counter clockwise Hence, θ =

–45°

Rotation about a point P (3, 2), so, first translate the object to origin then rotate the object and then translate back the rotated object to the given point P (3, 2).

$$\Delta A'B'C' = \Delta ABC.\,T.\,R.\,T^{-1}$$

$$\Delta A'B'C' = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & -2 & 1 \end{bmatrix} \begin{bmatrix} Cos\ 30° & Sin\ 30° & 0 \\ -Sin\ 30 & Cos\ 30° & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

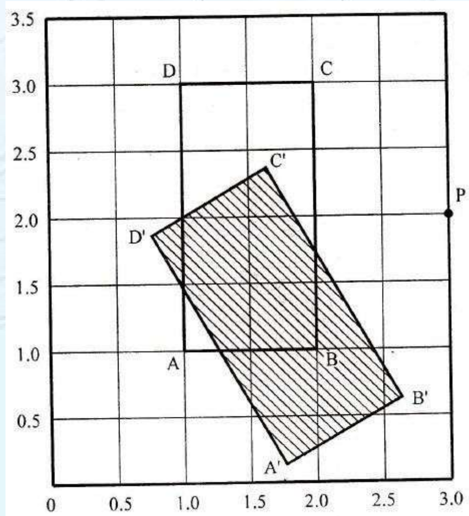
$$\Delta A'B'C' = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0.866 & 0.5 & 0 \\ -0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

$$\Delta A'B'C' = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0.866 & 0.5 & 0 \\ -0.5 & 0.866 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

$$\Delta A'B'C' = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 0.866 & 0.5 & 0 \\ -0.5 & 0.866 & 0 \\ 1.402 & -1.232 & 1 \end{bmatrix}$$

$$\Delta A'B'C' = \begin{bmatrix} 1.768 & 0.134 & 1 \\ 2.634 & 0.634 & 1 \\ 1.634 & 2.366 & 1 \\ 0.768 & 1.866 & 1 \end{bmatrix}$$

Answer: New vertices are A’ (1.768, 0.134), B’ (2.634, 0.634), C’ (1.634, 2.366) and D’ (0.768, 1.866).



Ex. 4.4 ***[GTU; Nov-2011; 7 Marks]***

A triangle PQR with vertices at P (0, 0), Q (4, 0) and R (2, 3) is given. Perform the following operations for it.

(i) Translation through 4 and 2 units along X and Y direction respectively,

(ii) Rotation through 90° in counter clockwise direction about the new position of point R.

Solution: (i) Translation by $t_x = 4$ and $t_y = 2$.

$$\Delta P'Q'R' = \Delta PQR.\,T$$

$$\Delta P'Q'R' = \begin{bmatrix} 0 & 0 & 1 \\ 4 & 0 & 1 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix}$$

$$\Delta P'Q'R' = \begin{bmatrix} 4 & 2 & 1 \\ 8 & 2 & 1 \\ 6 & 5 & 1 \end{bmatrix}$$

Answer: New vertices are P' (4, 2), Q' (8, 2) and R' (6, 5).

(ii) Rotation about the new point R' (6, 5) through 90° counter-clockwise. Hence,

$\theta = 90°$

Rotation about a point R' (6, 5), so, first translate the object to origin then rotate the object and then translate back the rotated object to the given point R' (6, 5).

$$\Delta P''Q'R'' = \Delta P'Q'R'.\,T.\,R.\,T^{-1}$$

$$\Delta P''Q''R' = \begin{bmatrix} 4 & 2 & 1 \\ 8 & 2 & 1 \\ 6 & 5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -6 & -5 & 1 \end{bmatrix} \begin{bmatrix} Cos\,90° & Sin\,90° & 0 \\ -Sin\,90 & Cos\,90° & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 6 & 5 & 1 \end{bmatrix}$$

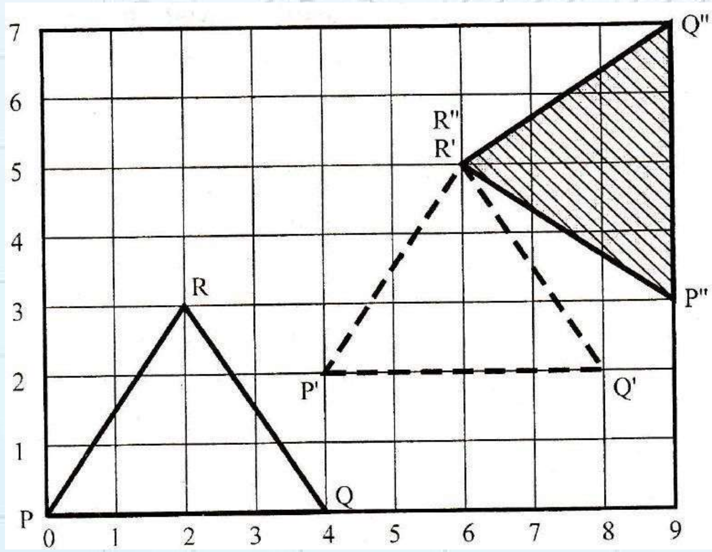
$$\Delta P''Q'R'' = \begin{bmatrix} 4 & 2 & 1 \\ 8 & 2 & 1 \\ 6 & 5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -6 & -5 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 6 & 5 & 1 \end{bmatrix}$$

$$\Delta P''Q'R'' = \begin{bmatrix} 4 & 2 & 1 \\ 8 & 2 & 1 \\ 6 & 5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -6 & -5 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 6 & 5 & 1 \end{bmatrix}$$

$$\Delta P''Q''R'' = \begin{bmatrix} 4 & 2 & 1 \\ 8 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$\Delta P''Q''R'' = \begin{bmatrix} 9 & 3 & 1 \\ 9 & 7 & 1 \\ 6 & 5 & 1 \end{bmatrix}$$

Answer: New vertices are P''(9, 3), Q''(9, 7) and R''(6, 5).



Ex. 4.5 ***[GTU; Nov-2011; 7 Marks]***

Reflect the diamond shape polygon whose vertices are A (−2, 0), B (0, −1) C (2, 0) and D (0, 1) about an arbitrary line L which is represented by equation y = 0.5 x + 1.

Solution: Data given:

Reflect the polygon about line y = 0.5 x + 1

So, first translate the object to origin, and then rotate the object such that the line coincides with x-axis, then reflect the polygon about x-axis, then rotate back and then translate back.

y = 0.5 x + 1

$m = 0.5 = \tan\theta$

$\theta = \tan^{-1} 0.5 = 26.565° \ (Clockwise)$

$\theta = -26.565°$

Also, when x = 0, y = 1 and when y = 0, x = -2.

$\therefore t_x = 2, \qquad t_y = 0$

$$A'B'C'D' = ABCD.T.R.M.R^{-1}T^{-1}$$

$$= \begin{bmatrix} -2 & 0 & 1 \\ 0 & -1 & 1 \\ 2 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ t_x & t_y & 1 \end{bmatrix} \begin{bmatrix} Cos\,\theta & Sin\,\theta & 0 \\ -Sin\,\theta & Cos\,\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} Cos\,(-\theta) & Sin\,(-\theta) & 0 \\ -Sin\,(-\theta) & Cos\,(-\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ t_x & t_y & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 & 1 \\ 0 & -1 & 1 \\ 2 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} Cos(-26.5) & Sin(-26.5) & 0 \\ -Sin(-26.5) & Cos(-26.5) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} Cos26.5 & Sin26.5 & 0 \\ -Sin26.5 & Cos26.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 & 1 \\ 0 & -1 & 1 \\ 2 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.8944 & -0.4472 & 0 \\ 0.4472 & 0.8944 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.8944 & 0.4472 & 0 \\ -0.4472 & 0.8944 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 & 1 \\ 0 & -1 & 1 \\ 2 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.8944 & -0.4472 & 0 \\ 0.4472 & 0.8944 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.8944 & 0.4472 & 0 \\ -0.4472 & 0.8944 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 & 1 \\ 0 & -1 & 1 \\ 2 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.8944 & -0.4472 & 0 \\ 0.4472 & 0.8944 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.8944 & 0.4472 & 0 \\ 0.4472 & -0.8944 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 & 1 \\ 0 & -1 & 1 \\ 2 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.6 & 0.8 & 0 \\ 0.8 & -0.6 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 & 1 \\ 0 & -1 & 1 \\ 2 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0.6 & 0.8 & 0 \\ 0.8 & -0.6 & 0 \\ -0.8 & 1.6 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 & 1 \\ -1.6 & 2.2 & 1 \\ 0.4 & 3.2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Answer: New vertices are A' (-2, 0), B' (-1.6, 2.2), C' (0.4, 3.2) and D' (0, 1).

4.6 References

1) CAD/CAM Principles and Applications by P. N. Rao, Tata McGraw-Hill Publication.
2) Mastering CAD/CAM by Zeid,Ibrahim, Tata McGraw-Hill Publication.
3) CAD/CAM/CIM by Radhakrishnan,P;Subramanyan,S;Raju,V, New Age International (P) Limited.
4) CAD/CAM: Theory and Practice by Zeid,Ibrahim;Subramanian,R, Tata McGraw-Hill Publication.

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