

UNIT – 6

OPTIMIZATION

- Important Repeated Questions

- 6.1 Definition
- 6.2 Conventional Versus Optimum Design Process
- 6.3 Engineering Applications of Optimization
- 6.4 Statement of an Optimization Problem
- 6.5 Terms used in Optimization
- 6.6 Classification of Optimization Problem
- 6.7 Optimum Design with Lagrange's multipliers
- 6.8 Johnson's Method of Optimum Design

Important Repeated Questions:

1. **Define Finite Element Analysis (FEA) and explain its application/significance in engineering.** (W25 - Q4a, 03 marks) (S25 - Q4a, 03 marks)
2. **Explain the Penalty approach for handling boundary conditions in FEA with an example.** (W25 - Q4c, 07 marks) (S25 - Q4b, 04 marks) (S22 - Q5b OR, 04 marks) (W23 - Q5a, 03 marks)
3. **Determine nodal displacements, stresses, and reaction forces for a given 1-D bar/stepped bar/stepped shaft using FEA (using Penalty or Elimination approach).** (W25 - Q4c, 07 marks) (S23 - Q5c, 07 marks) (S23 - Q5c OR, 07 marks) (S24 - Q4c, 07 marks) (S24 - Q4c OR, 07 marks) (W22 - Q5c, 07 marks) (W23 - Q4c, 07 marks) (S22 - Q5c, 07 marks) (S22 - Q5c OR, 07 marks) (S25 - Q4c, 07 marks) (W24 - Q4c OR, 07 marks)
4. **What is discretization? Explain its concept and importance in FEA.** (W25 - Q3b OR, 04 marks) (S24 - Q4a OR, 03 marks) (S25 - Q4a OR, N/A marks)
5. **Explain the General/Basic steps involved in solving a problem using FEA.** (W25 - Q3a OR, 03 marks) (W23 - Q4b, N/A marks)
6. **Explain/Discuss the properties of the global stiffness matrix.** (W22 - Q5a, 03 marks) (W23 - Q5b, 04 marks) (S22 - Q5a OR, 03 marks) (W24 - Q4a OR, 03 marks)
7. **Define/Optimization in the context of engineering and explain its importance/applications.** (W25 - Q5a, 03 marks) (S24 - Q5b, 04 marks) (W23 - Q4a OR, 03 marks) (S25 - Q5a, 03 marks) (W24 - Q4c OR, 07 marks)
8. **Explain Johnson's method of optimum design.** (W25 - Q5c OR, 07 marks) (W23 - Q5c, 07 marks) (S25 - Q5c, N/A marks)
9. **Explain/Differentiate between constrained and unconstrained optimization.** (W25 - Q5c, 07 marks)
10. **Explain the following with reference to optimization: i) Objective function ii) Constraints.** (W22 - Q4c, 07 marks) (W24 - Q4b, 04 marks)
11. **Explain how the stiffness matrix is derived from the strain-displacement matrix.** (W25 - Q5b OR, 04 marks) (S25 - Q5b, 04 marks)
12. **Explain/Differentiate between Plane stress and Plane strain conditions.** (S23 - Q5b, 04 marks) (W23 - Q4b OR, 04 marks) (W24 - Q4b OR, 04 marks)
13. **Explain the elimination approach used in FEA.** (W22 - Q5b, 04 marks)

Legends: W- Winter, S- Summer, Q- Question and 03/04/07- Marks of Question

6.1 Definition

- Optimization is the act of obtaining the best result under given circumstances.
- In design, construction, and maintenance of any engineering system, engineers have to take many technological and managerial decisions at several stages. The ultimate goal of all such decisions is either to minimize the effort required or to maximize the desired benefit.
- *Optimization* can be defined as the process of finding the conditions that give the maximum or minimum value of a function.

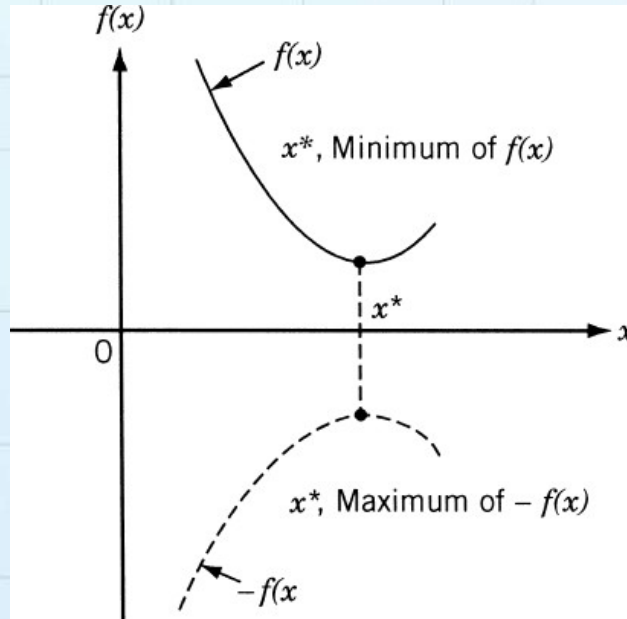


Fig. 6.1 Minimum of $f(x)$ is same as maximum of $-f(x)$.

- The optimum seeking methods are also known as *mathematical programming techniques* and are generally studied as a part of operations research.
- *Operations research* is a branch of mathematics concerned with the application of scientific methods and techniques to decision making problems and with establishing the best or optimal solutions.

6.2 Conventional Versus Optimum Design Process

- It is a challenge for engineers to design efficient and cost-effective systems without compromising their integrity. Fig. 6.2 (a) presents a self-explanatory flowchart for a conventional design method; Fig. 6.2(b) presents a similar flowchart for the optimum design method.
- It is important to note that both methods are iterative, as indicated by a loop between blocks 6 and 3. Both methods have some blocks that require similar calculations and others that require different calculations. The key features of the two processes are these:

1. The optimum design method has block 0, where the problem is formulated as one of optimization. An objective function is defined that measures the merits of different designs.
2. Both methods require data to describe the system in block 1.
3. Both methods require an initial design estimate in block 2.
4. Both methods require analysis of the system in block 3.
5. In block 4, the conventional design method checks to ensure that the performance criteria are met, whereas the optimum design method checks for satisfaction of all of the constraints for the problem formulated in block 0.
6. In block 5, stopping criteria for the two methods are checked, and the iteration is stopped if the specified stopping criteria are met.
7. In block 6, the conventional design method updates the design based on the designer's experience and intuition and other information gathered from one or more trial designs; the optimum design method uses optimization concepts and procedures to update the current design.

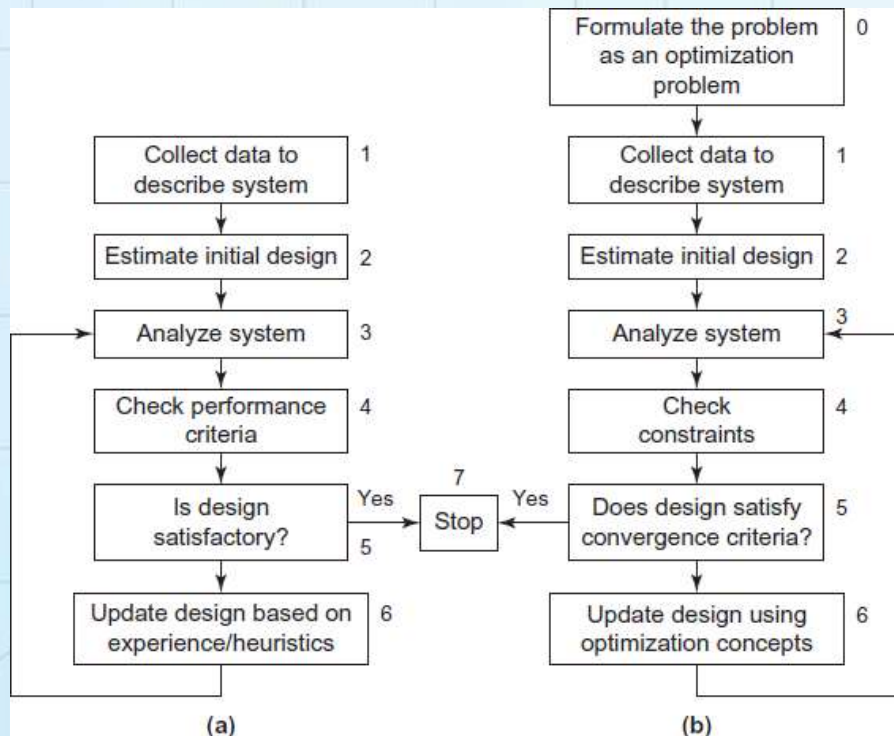


Fig. 6.2 Comparison of (a) conventional design method and (b) optimum design method.

- The foregoing distinction between the two design approaches indicates that the conventional design process is less formal. An objective function that measures a design's merit is not identified. Trend information is usually not calculated; nor is it used in block 6 to make design decisions for system improvement. In contrast, the optimization process is more formal, using trend information to make design changes.

6.3 Engineering Applications of Optimization

Optimization, in its broadest sense, can be applied to solve any engineering problem. Some typical applications from different engineering disciplines indicate the wide scope of the subject:

1. Design of aircraft and aerospace structures for minimum weight
2. Finding the optimal trajectories of space vehicles
3. Design of civil engineering structures such as frames, foundations, bridges, towers, chimneys, and dams for minimum cost
4. Minimum-weight design of structures for earthquake, wind, and other types of random loading
5. Design of water resources systems for maximum benefit
6. Optimal plastic design of structures
7. Optimum design of linkages, cams, gears, machine tools, and other mechanical components
8. Selection of machining conditions in metal-cutting processes for minimum production cost
9. Design of material handling equipment, such as conveyors, trucks, and cranes, for minimum cost
10. Design of pumps, turbines, and heat transfer equipment for maximum efficiency
11. Optimum design of electrical machinery such as motors, generators, and transformers
12. Optimum design of electrical networks
13. Shortest route taken by a salesperson visiting various cities during one tour
14. Optimal production planning, controlling, and scheduling
15. Analysis of statistical data and building empirical models from experimental results to obtain the most accurate representation of the physical phenomenon
16. Optimum design of chemical processing equipment and plants
17. Design of optimum pipeline networks for process industries
18. Selection of a site for an industry
19. Planning of maintenance and replacement of equipment to reduce operating costs
20. Inventory control
21. Allocation of resources or services among several activities to maximize the benefit
22. Controlling the waiting and idle times and queueing in production lines to reduce the costs
23. Planning the best strategy to obtain maximum profit in the presence of a competitor
24. Optimum design of control systems.

6.4 Statement of an Optimization Problem

An optimization or a mathematical programming problem can be stated as follows.

Find $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ which minimizes $f(X)$

$$\begin{bmatrix} 1 \\ \vdots \\ n \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ J \end{bmatrix}$$

Subject to the constraints

$$g_j(X) \leq 0, \quad j = 1, 2, \dots, m$$

$$l_j(X) = 0, \quad j = 1, 2, \dots, p \quad (6.1)$$

where X is an n -dimensional vector called the *design vector*,

$f(X)$ is termed the *objective function*, and

$g_j(X)$ and $l_j(X)$ are known as *inequality* and *equality* constraints, respectively.

The problem stated in Eq. (6.1) is called a *constrained optimization problem*.

Some optimization problems do not involve any constraints and can be stated as

Find $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ which minimizes $f(X)$

$$\begin{bmatrix} 1 \\ \vdots \\ n \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ J \end{bmatrix}$$

Such problems are called unconstrained optimization problems.

6.5 Terms used in optimization

I. Design Vector and preassigned parameters.

- Any engineering system or component is defined by a set of quantities some of which are viewed as variables during the design process.
- In general, certain quantities are usually fixed at the outset and these are called *preassigned parameters*.
- All the other quantities are treated as variables in the design process and are called *design* or *decision variables*. The design variables are collectively represented as a design vector.
- As an example, consider the design of the gear pair shown in Fig. 6.3, characterized by its face width b , number of teeth T_1 and T_2 , center distance d , pressure angle ϕ , tooth profile, and material.
- If center distance, pressure angle ϕ , tooth profile, and material of the gears are fixed in advance, these quantities can be called *preassigned parameters*.
- The remaining quantities can be collectively represented by

– A design Vector $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b \\ \phi \\ T_1 \\ T_2 \end{bmatrix}$

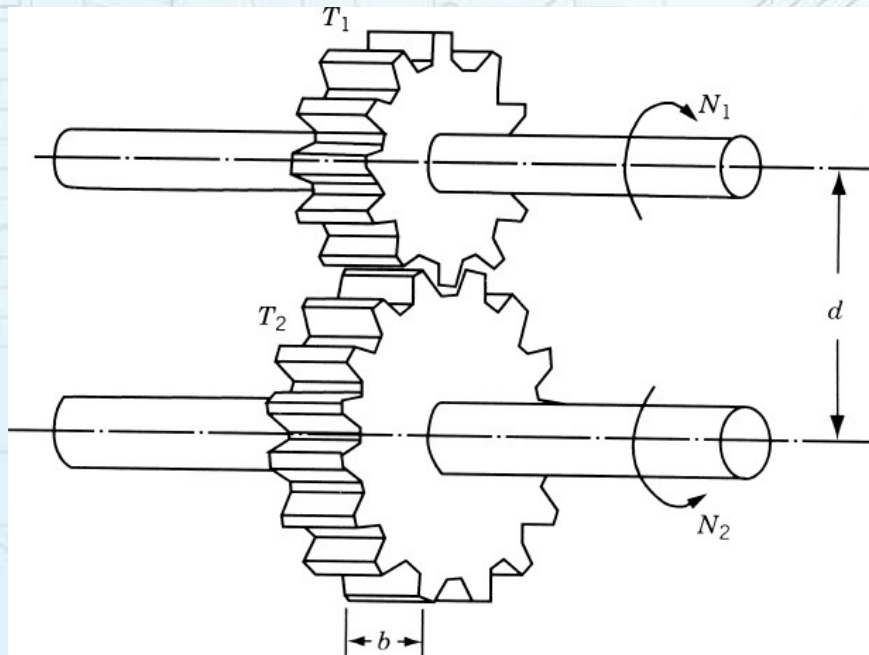


Fig. 6.3 Gear pair in mesh.

II. Design Space or Design Variable Space

- If an n-dimensional Cartesian space with each coordinate axis representing a design variable x_i ($i = 1, 2, \dots, n$) is considered, the space is called the *design variable space* or simply *design space*. Each point in the n-dimensional design space is called a *design point* and represents either a possible or an impossible solution to the design problem.
- In the case of the design of a gear pair, the design point $\{1.0, 20, 40\}^T$, for example, represents a possible solution, whereas the design point $\{1.0, -20, 40.5\}^T$ represents an impossible solution since it is not possible to have either a negative value or a fractional value for the number of teeth.

III. Design Constraints, Behavior Constraints and Side Constraints

- In many practical problems, the design variables cannot be chosen arbitrarily; rather, they have to satisfy certain specified functional and other requirements. The restrictions that must be satisfied to produce an acceptable design are collectively called *design constraints*.
- Constraints that represent limitations on the behavior or performance of the system are termed *behavior* or *functional constraints*.
- Constraints that represent physical limitations on design variables, such as availability, fabricability, and transportability, are known as *geometric* or *side constraints*.
- For example, for the gear pair shown in Fig. 6.2, the face width b cannot be taken smaller than a certain value, due to strength requirements. Similarly, the ratio of the numbers of teeth, T_1/T_2 , is dictated by the speeds of the input and output shafts, N_1

and N_2 . Since these constraints depend on the performance of the gear pair, they are called behavior constraints.

- The values of T_1 and T_2 cannot be any real numbers but can only be integers. Further, there can be upper and lower bounds on T_1 and T_2 due to manufacturing limitations. Since these constraints depend on the physical limitations, they are called *side constraints*.

IV. Constraint Surface

- For illustration, consider an optimization problem with only inequality constraints $g_j(\mathbf{X}) \leq 0$. The set of values of \mathbf{X} that satisfy the equation $g_j(\mathbf{X}) = 0$ forms a hypersurface in the design space and is called a *constraint surface*.
- The constraint surface divides the design space into two regions:
 - one in which $g_j(\mathbf{X}) < 0$ and the other in which $g_j(\mathbf{X}) > 0$.
- Thus the points lying on the hypersurface will satisfy the constraint $g_j(\mathbf{X})$ critically, whereas the points lying in the region where $g_j(\mathbf{X}) > 0$ are infeasible or unacceptable, and the points lying in the region where $g_j(\mathbf{X}) < 0$ are feasible or acceptable. The collection of all the constraint surfaces $g_j(\mathbf{X}) = 0, j = 1, 2, \dots, m$, which separates the acceptable region is called the *composite constraint surface*.

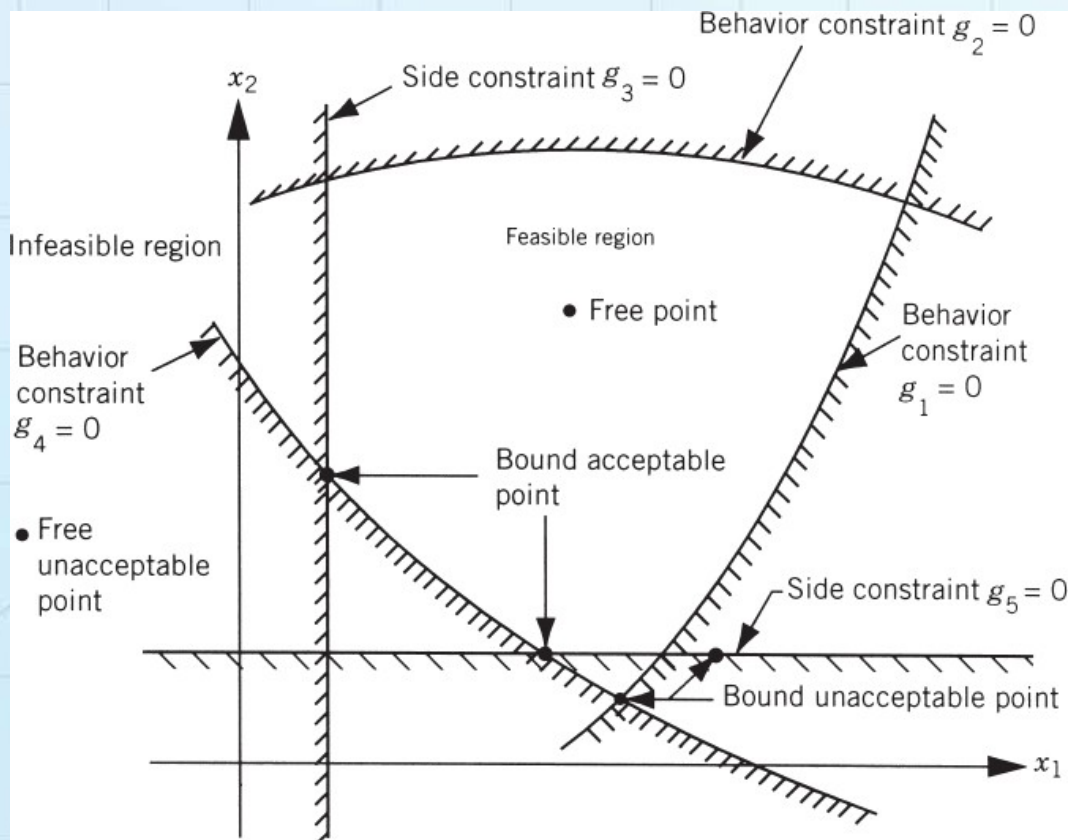


Fig. 6.4 Constraint surfaces in a hypothetical two-dimensional design space.

- Fig.6.4 shows a hypothetical two-dimensional design space where the infeasible region is indicated by hatched lines. A design point that lies on one or more than one constraint surface is called a *bound point*, and the associated constraint is called an *active constraint*.
- Design points that do not lie on any constraint surface are known as *free points*.
- Depending on whether a particular design point belongs to the acceptable or unacceptable region, it can be identified as one of the following four types:
 1. Free and acceptable point
 2. Free and unacceptable point
 3. Bound and acceptable point
 4. Bound and unacceptable point

V. Objective Function

- The criterion, with respect to which the design is optimized, when expressed as a function of the design variables, is known as the *criterion* or *merit* or *objective function*.
- The choice of objective function is governed by the nature of problem. The objective function for minimization is generally taken as weight in aircraft and aerospace structural design problems.
- In civil engineering structural designs, the objective is usually taken as the minimization of cost.
- The maximization of mechanical efficiency is the obvious choice of an objective in mechanical engineering systems design.
- In some situations, there may be more than one criterion to be satisfied simultaneously.
- For example, a gear pair may have to be designed for minimum weight and maximum efficiency while transmitting a specified horsepower.
- An optimization problem involving multiple objective functions is known as a *multiobjective programming problem*.
- If $f_1(\mathbf{X})$ and $f_2(\mathbf{X})$ denote two objective functions, construct a new (overall) objective function for optimization as

$$f(\mathbf{X}) = a_1 f_1(\mathbf{X}) + a_2 f_2(\mathbf{X})$$

- where a_1 and a_2 are constants whose values indicate the relative importance of one objective function relative to the other.

6.6 Classification of Optimization Problems

Optimization problems can be classified in several ways, as described below.

(a) Classification Based on the Existence of Constraints

- Any optimization problem can be classified as constrained or unconstrained, depending on whether constraints exist in the problem.

(b) Classification Based on the Nature of the Design Variables

- Based on the nature of design variables encountered, optimization problems can be classified into two broad categories. In the first category, the problem is to find values to a set of design parameters that make some prescribed function of these parameters minimum subject to certain constraints. For example, the problem of minimum-weight design of a prismatic beam shown in Fig. 6.9a subject to a limitation on the maximum deflection can be stated as follows:

$$\text{Find } X = \begin{bmatrix} b \\ d \end{bmatrix} \text{ which minimizes}$$

$$f(X) = plbd$$

subject to the constraints

$$J_{tip}(X) \leq J_{max}$$

$$b \geq 0 \text{ and } d \geq 0$$

- Where p is the density and J_{tip} is the tip deflection of the beam. Such problems are called *parameter* or *static optimization problems*.

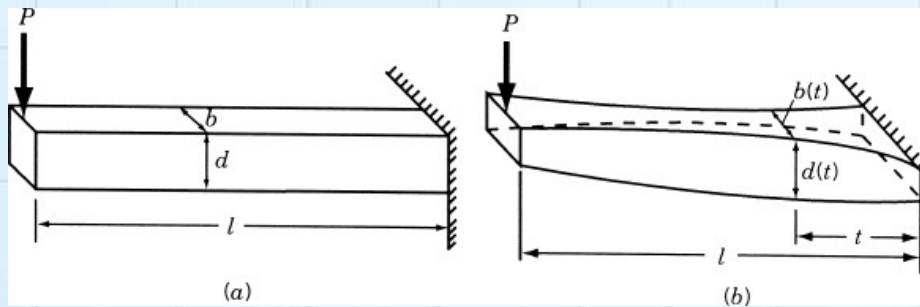


Fig. 6.5 Cantilever beam under concentrated load.

- In the second category of problems, the objective is to find a set of design parameters, which are all continuous functions of some other parameter, that minimizes an objective function subject to a set of constraints. If the cross-sectional dimensions of the rectangular beam are allowed to vary along its length as shown in Fig. 6.9b, the optimization problem can be stated as

$$\text{Find } \mathbf{X}(t) = \begin{bmatrix} b(t) \\ d(t) \end{bmatrix} \text{ which minimizes}$$

$$f[\mathbf{X}(t)] = \rho \int_0^l b(t) d(t) dt$$

subject to the constraints

$$\delta_{tip}[\mathbf{X}(t)] \leq \delta_{max}, \quad 0 \leq t \leq l$$

$$b(t) \geq 0, \quad 0 \leq t \leq l$$

$$d(t) \geq 0, \quad 0 \leq t \leq l$$

- Here the design variables are functions of the length parameter t . This type of problem, where each design variable is a function of one or more parameters, is known as a *trajectory* or *dynamic optimization problem*.

(c) Classification Based on the Physical Structure of the Problem

- Depending on the physical structure of the problem, optimization problems can be classified as optimal control and nonoptimal control problems.
- **Optimal Control Problem.** An *optimal control (OC) problem* is a mathematical programming problem involving a number of stages, where each stage evolves from the preceding stage in a prescribed manner.
- It is usually described by two types of variables the control (design) and the state variables. The *control variables* define the system and govern the evolution of the system from one stage to the next, and the *state variables* describe the behavior or status of the system in any stage.

(d) Classification Based on the Nature of the Equations Involved

- Another important classification of optimization problems is based on the nature of expressions for the objective function and the constraints. According to this classification, optimization problems can be classified as linear, nonlinear, geometric, and quadratic programming problems. This classification is extremely useful from the computational point of view since there are many special methods available for the efficient solution of a particular class of problems. Thus the first task of a designer would be to investigate the class of problem encountered. This will, in many cases, dictate the types of solution procedures to be adopted in solving the problem.
- **Nonlinear Programming Problem.** If any of the functions among the objective and constraint functions in Eq. (1.1) is nonlinear, the problem is called a *nonlinear programming (NLP) problem*. This is the most general programming problem and all other problems can be considered as special cases of the NLP problem.
- **Linear Programming Problem.** If the objective function and all the constraints are linear functions of the design variables, the mathematical programming problem is called a *linear programming (LP) problem*. A linear programming problem is often stated in the following standard form:

$$\text{Find } \mathbf{X} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix}$$

$$\text{which minimizes } f(\mathbf{X}) = \sum_{i=1}^n c_i x_i$$

subject to the constraints

$$\sum_{i=1}^n a_{ij} x_i = b_j, \quad j = 1, 2, \dots, m$$

$$x_i \geq 0, \quad i = 1, 2, \dots, n$$

where c_i , a_{ij} , and b_j are constants.

– **Geometric Programming Problem.**

– **Definition :** A function $h(\mathbf{X})$ is called a *posynomial* if h can be expressed as the sum of power terms each of the form

$$c_i x_1^{a_{i1}} x_2^{a_{i2}} \dots x_n^{a_{in}}$$

– where c_i and a_{ij} are constants with $c_i > 0$ and $x_j > 0$. Thus a posynomial with N terms can be expressed as

$$h(\mathbf{X}) = c_1 x_1^{a_{11}} x_2^{a_{12}} \dots x_n^{a_{1n}} + \dots + c_N x_1^{a_{N1}} x_2^{a_{N2}} \dots x_n^{a_{Nn}}$$

– A *geometric programming (GMP) problem* is one in which the objective function and constraints are expressed as posynomials in \mathbf{X} .

– **Quadratic Programming Problem.** A quadratic programming problem is a nonlinear programming problem with a quadratic objective function and linear constraints. It is usually formulated as follows:

$$F(\mathbf{X}) = c + \sum_{i=1}^n q_i x_i + \sum_{i=1}^n \sum_{j=1}^n Q_{ij} x_i x_j$$

subject to

$$\sum_{i=1}^n a_{ij} x_i = b_j, \quad j = 1, 2, \dots, m$$

$$x_i \geq 0, \quad i = 1, 2, \dots, n$$

where c, q_i, Q_{ij}, a_{ij} , and b_j are constants.

(e) Classification Based on the Permissible Values of the Design Variables

– Depending on the values permitted for the design variables, optimization problems can be classified as integer and real-valued programming problems.

– **Integer Programming Problem.** If some or all of the design variables x_1, x_2, \dots, x_n of an optimization problem are restricted to take on only integer (or discrete) values, the problem is called an *integer programming problem*.

– On the other hand, if all the design variables are permitted to take any real value, the optimization problem is called a *real-valued programming problem*.

(f) Classification Based on the Deterministic Nature of the Variables

– Based on the deterministic nature of the variables involved, optimization problems can be classified as deterministic and stochastic programming problems.

– **Stochastic Programming Problem.** A stochastic programming problem is an optimization problem in which some or all of the parameters (design variables and/or preassigned parameters) are probabilistic (nondeterministic or stochastic).

(g) Classification Based on the Separability of the Functions

– Optimization problems can be classified as separable and nonseparable programming problems based on the separability of the objective and constraint functions.

- **Separable Programming Problem.**
- **Definition:** A function $f(\mathbf{X})$ is said to be *separable* if it can be expressed as the sum of n single-variable functions, $f_1(x_1), f_2(x_2), \dots, f_n(x_n)$, that is,
- A separable programming problem is one in which the objective function and the constraints are separable and can be expressed in standard form as

$$\text{Find } \mathbf{X} \text{ which minimizes } f(\mathbf{X}) = \sum_{i=1}^n f_i(x_i)$$

$$\text{Subject to } g_j(\mathbf{X}) = \sum_{i=1}^n g_{ij}(x_i) \leq b_j \quad j = 1, 2, \dots, m$$

Where b_j is a constant

(h) Classification Based on the Number of Objective Functions

Depending on the number of objective functions to be minimized, optimization problems can be classified as single and multiobjective programming problems.

Multiobjective Programming Problem. A multiobjective programming problem can be stated as follows:

$$\text{Find } \mathbf{X} \text{ which minimizes } f_1(\mathbf{X}), f_2(\mathbf{X}), \dots, f_k(\mathbf{X})$$

subject to

$$g_j(\mathbf{X}) \leq 0, j = 1, 2, \dots, m$$

where f_1, f_2, \dots, f_k denote the objective functions to be minimized simultaneously.

Example 6.1:

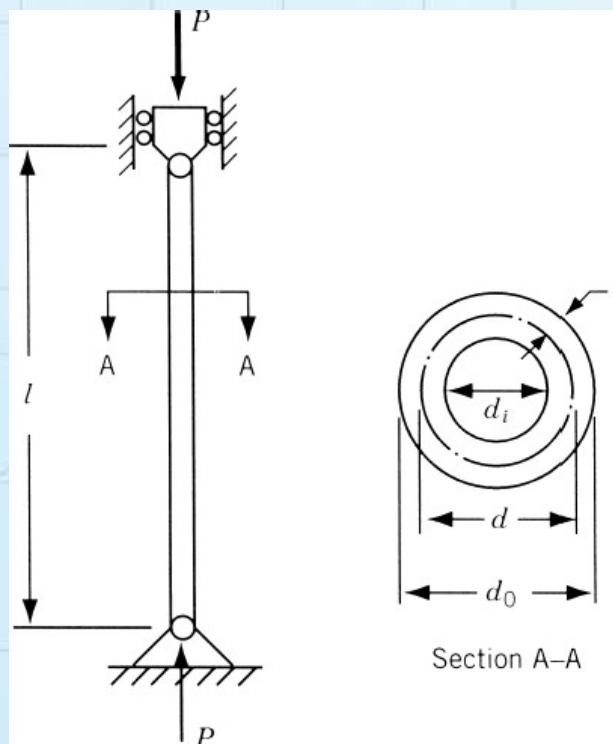


Fig. 6.6 Tubular column under compression.

Design a uniform column of tubular section, with hinge joints at both ends, (Fig. 6.4) to carry a compressive load $P = 2500$ kgf for minimum cost. The column is made up of a material that has a yield stress (σ_y) of 500 kgf/cm², modulus of elasticity (E) of 0.85×10^6 kgf/cm², and weight density (ρ) of 0.0025 kgf/cm³. The length of the column is 250 cm. The stress induced in the column should be less than the buckling stress as well as the yield stress. The mean diameter of the column is restricted to lie between 2 and 14 cm, and columns with thicknesses outside the range 0.2 to 0.8 cm are not available in the market. The cost of the column includes material and construction costs and can be taken as $5W + 2d$, where W is the weight in kilograms force and d is the mean diameter of the column in centimeters.

The design variables are the mean diameter (d) and tube thickness (t):

$$\mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} d \\ t \end{bmatrix}$$

The objective function to be minimized is given by

$$\begin{aligned} f(\mathbf{X}) &= 5W + 2d \\ &= 5(\rho V) + 2d \\ &= 5(\rho A l) + 2d \\ &= 5(\rho \pi d t l) + 2d \\ &= 9.82x_1x_2 + 2x_1 \end{aligned}$$

The behavior constraints can be expressed as

$$\begin{aligned} \text{stress induced} &\leq \text{yield stress} \\ \text{stress induced} &\leq \text{buckling stress} \end{aligned}$$

The induced stress is given by

$$\text{induced stress} = \sigma_i = \frac{P}{\pi dt} = \frac{2500}{\pi x_1 x_2}$$

The buckling stress for a pin-connected column is given by

$$\text{buckling stress} = \sigma_b = \frac{\text{Euler buckling load}}{\text{cross-sectional area}} = \frac{\pi^2 EI}{l^2} \frac{1}{\pi dt}$$

where

I = second moment of area of the cross section of the column

$$\begin{aligned} &= \frac{\pi}{64}(d_o^4 - d_i^4) \\ &= \frac{\pi}{64}(d_o^2 + d_i^2)(d_o + d_i)(d_o - d_i) = \frac{\pi}{64}[(d+t)^2 + (d-t)^2] \\ &\quad \times [(d+t) + (d-t)][(d+t) - (d-t)] \\ &= \frac{\pi}{8} dt(d^2 + t^2) = \frac{\pi}{8} x_1 x_2 (x_1^2 + x_2^2) \end{aligned}$$

Thus the behavior constraints can be restated as

$$g_1(\mathbf{X}) = \frac{2500}{\pi x_1 x_2} - 500 \leq 0$$

$$g_2(\mathbf{X}) = \frac{2500}{\pi x_1 x_2} - \frac{\pi^2 (0.85 \times 10^6) (x_1^2 + x_2^2)}{8(250)^2} \leq 0$$

The side constraints are given by

$$2 \leq d \leq 14$$

$$0.2 \leq t \leq 0.8$$

which can be expressed in standard form as

$$g_3(\mathbf{X}) = -x_1 + 2.0 \leq 0$$

$$g_4(\mathbf{X}) = x_1 - 14.0 \leq 0$$

$$g_5(\mathbf{X}) = -x_2 + 0.2 \leq 0$$

$$g_6(\mathbf{X}) = x_2 - 0.8 \leq 0$$

Since there are only two design variables, the problem can be solved graphically as shown below.

First, the constraint surfaces are to be plotted in a two-dimensional design space where the two axes represent the two design variables x_1 and x_2 . To plot the first constraint surface, we have

$$g_1(\mathbf{X}) = \frac{2500}{\pi x_1 x_2} - 500 \leq 0$$

that is, $x_1 x_2 \geq 1.593$

Thus the curve $x_1 x_2 = 1.593$ represents the constraint surface $g_1(\mathbf{X}) = 0$. This curve can be plotted by finding several points on the curve. The points on the curve can be found by giving a series of values to x_1 and finding the corresponding values of x_2 that satisfy the relation $x_1 x_2 = 1.593$:

x_1	2.0	4.0	6.0	8.0	10.0	12.0	14.0
x_2	0.7965	0.3983	0.2655	0.1990	0.1593	0.1328	0.1140

These points are plotted and a curve P_1Q_1 passing through all these points is drawn as shown in Fig. 6.7, and the infeasible region, represented by $g_1(\mathbf{X}) > 0$ or $x_1 x_2 < 1.593$, is shown by hatched lines. Similarly, the second constraint $g_2(\mathbf{X}) \leq 0$ can be expressed as $x_1 x_2 (x_1^2 + x_2^2) \geq 47.3$ and the points lying on the constraint surface $g_2(\mathbf{X}) = 0$ can be obtained as follows for $x_1 x_2 (x_1^2 + x_2^2) = 47.3$:

x_1	2	4	6	8	10	12	14
x_2	2.41	0.716	0.219	0.0926	0.0473	0.0274	0.0172

These points are plotted as curve P_2Q_2 , the feasible region is identified, and the infeasible region is shown by hatched lines as in Fig. 6.7. The plotting of side constraints is very simple since they represent straight lines. After plotting all the six constraints, the feasible region can be seen to be given by the bounded area $ABCDEA$.

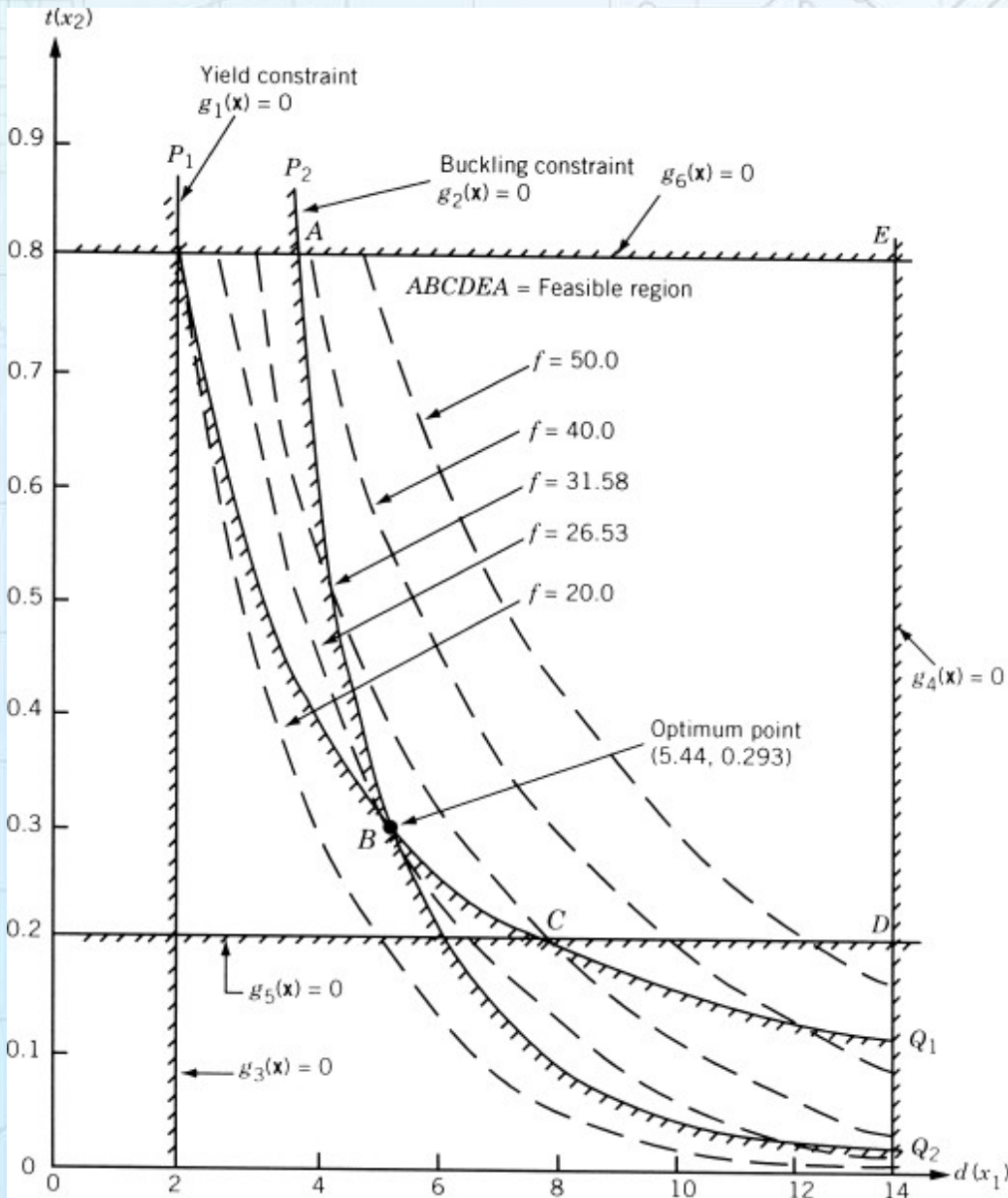


Fig. 6.7 Graphical optimization of Example 6.1.

Next, the contours of the objective function are to be plotted before finding the optimum point. For this, we plot the curves given by

$$f(\mathbf{X}) = 9.82x_1x_2 + 2x_1 = c = \text{constant}$$

for a series of values of c . By giving different values to c , the contours of f can be plotted with the help of the following points.

For 9.82 $x_1 x_2 + 2 x_1 = 50$

x_2	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
x_1	16.77	12.62	10.10	8.44	7.24	6.33	5.64	5.07

For 9.82 $x_1 x_2 + 2 x_1 = 40$

x_2	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
x_1	13.40	10.10	8.08	6.75	5.79	5.06	4.51	4.05

For 9.82 $x_1 x_2 + 2 x_1 = 31.58$ (Passing through corner point C)

x_2	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
x_1	10.57	7.96	6.38	5.33	4.57	4.00	3.56	3.20

For 9.82 $x_1 x_2 + 2 x_1 = 26.53$ (Passing through corner point B)

x_2	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
x_1	8.88	6.69	5.36	4.48	3.84	3.36	2.29	2.69

For 9.82 $x_1 x_2 + 2 x_1 = 20$

x_2	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
x_1	6.70	5.05	4.04	3.38	2.90	2.53	2.26	2.02

These contours are shown in Fig. 6.7 and it can be seen that the objective function cannot be reduced below a value of 26.53 (corresponding to point B) without violating some of the constraints.

Thus the optimum solution is given by point B with

$$d^* = x_1^* = 5.44 \text{ cm}$$

$$t^* = x_2^* = 0.293 \text{ cm}$$

$$f_{\min} = 26.53.$$

Example 6.2: It is required to design an oil can, shown in Figure 6.8, to hold at least 400 ml of oil ($1 \text{ ml} = 1 \text{ cm}^3$), as well as to meet other design requirements. The cans will be produced in the billions, so it is desirable to minimize their manufacturing costs. Since cost can be directly related to the surface area of the sheet metal used, it is reasonable to minimize the amount of sheet metal required. Fabrication, handling, aesthetics, and shipping considerations impose the following restrictions on the size of the can: The diameter should be no more than 8 cm and no less than 3.5 cm, whereas the height should be no more than 18 cm and no less than 8 cm.

The two design variables are defined as

D = diameter of the can, cm

H = height of the can, cm

Design vector $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} D \\ H \end{bmatrix}$

The objective function is minimization of cost of manufacturing of can

$$f(x) = 3.14x_1^2x_2 + 0.5x_1^2 \dots \dots \dots (1)$$

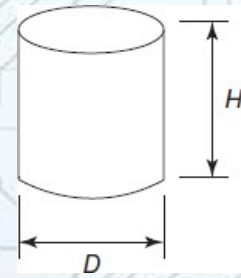


Fig. 6.8 Can

Side constraints

$$8 \leq D \leq 3.5 \text{ cm} \\ 18 \leq H \leq 8 \text{ cm} \dots \dots \dots (2)$$

Behavior constraints

$$V \geq 400 \text{ ml (or cm}^3\text{)} \\ \frac{\pi}{4} D^2 H \geq 400 \text{ cm}^3 \\ x_1^2 x_2 \geq 509.3 \dots \dots \dots (3) \\ g(x) = x_1^2 x_2 \geq 509.3$$

Get the solution from (2), (1) and (a)

x_1	3.5	4.5	6	7	8
x_2	41.57	25.15	14.14	10.39	8
$f(x)$	476.08	387.16	323.05	305.37	301.44

At $x_1 = 8$ and $x_2 = 8$ objective function is minimize and also satisfy the side constraints so, $D = 8 \text{ cm}$, $H = 8 \text{ cm}$.

To plot the graph and to find feasible region with optimum solution, find x_2 for various $f(x)$ by varying x_1 .

Let $f(x) = 3.14(x_1^2 x_2 + 0.5x_1^2) = 200$

x_1	1	2	3	4	5	6	7	8
x_2	63.19	30.84	19.73	14	10.24	7.61	5.6	3.96

Let $f(x) = 3.14(x_1^2 x_2 + 0.5x_1^2) = 300$

x_1	1	2	3	4	5	6	7	8
x_2	95.04	46.71	30.39	21.88	16.6	12.92	10.15	7.94

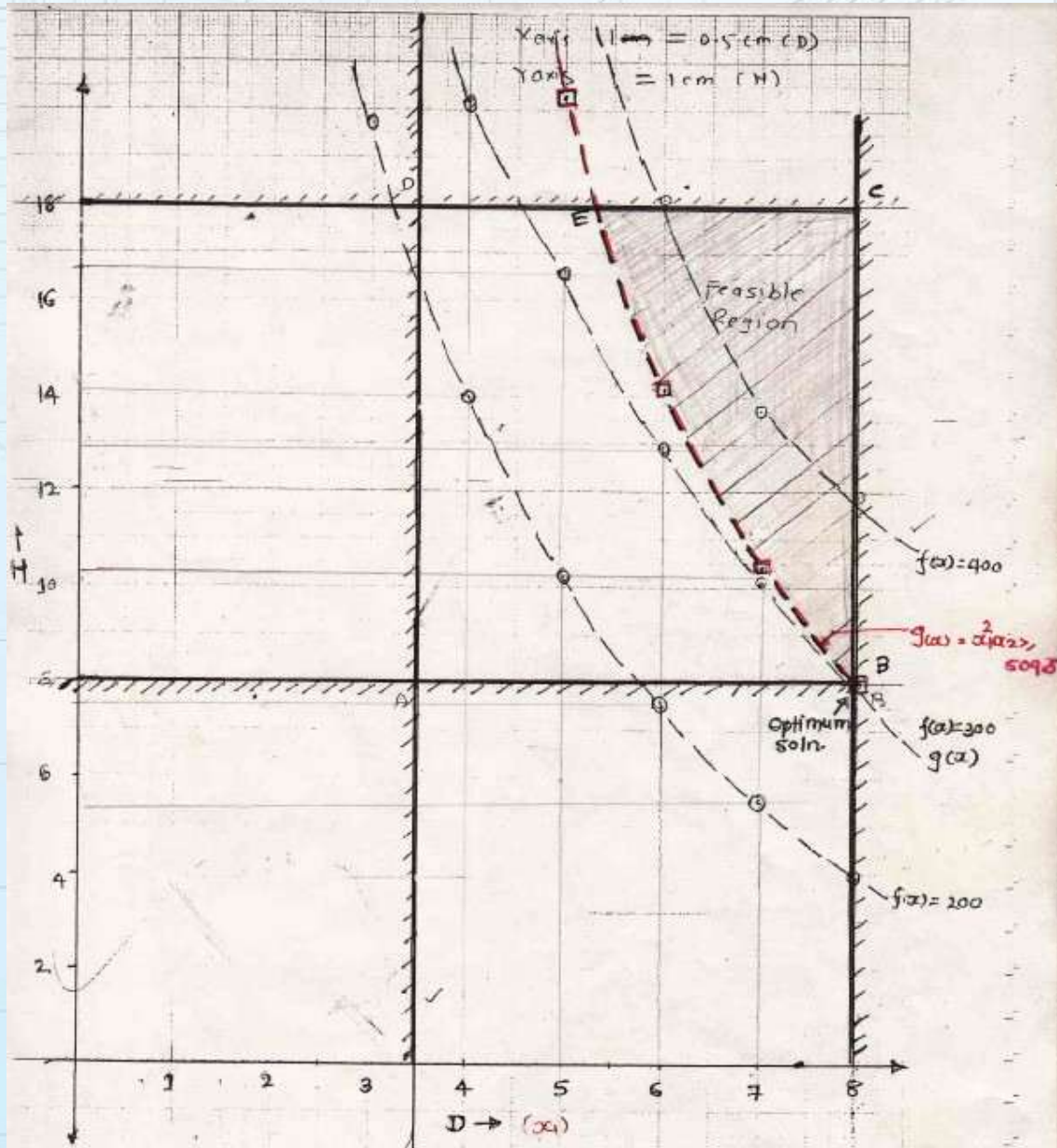


Fig. 6.9 Graphical optimization

Let $f(x) = 3.14(x_1 x_2 - 0.5x_1^2) = 400$

x_1	1	2	3	4	5	6	7	8
x_2	127.8	62.7	41	29.89	23	18.28	14.7	11.92

Then $g(x) = x_1^2 x_2 = 509.3$

x_1	1	2	3	4	5	6	7	8
x_2	509.3	127.3	56.58	31.83	20.37	14.14	10.4	8

Now plot $f(200), f(300), f(400)$ and $g(509.3)$

From the graph BCE is Feasible Region. Point B shows optimum solution which gives minimum surface area and minimum cost.

$$D = x_1 = 8 \text{ cm}$$

$$H = x_2 = 8 \text{ cm}$$

$$f(x) = 301.44$$

Example 6.3: A company manufactures two machines, A and B. Using available resources; either 28 A or 14 B can be manufactured daily. The sales department can sell up to 14 A machines or 24 B machines. The shipping facility can handle no more than 16 machines per day. The company makes a profit of \$400 on each A machine and \$600 on each B machine. How many A and B machines should the company manufacture every day to maximize its profit?

Design variable

The following two design variables are identified in the problem statement:

x_1 = number of A machines manufactured each day

x_2 = number of B machines manufactured each day

Objective function

The objective is to maximize daily profit, which can be expressed in terms of design variables as

$$f(x) = 400x_1 + 600x_2$$

Identification of constraints:

Design constraints are placed on manufacturing capacity, on sales personnel, and on the shipping and handling facility.

The constraint on the shipping and handling facility is quite straightforward:

$x_1 + x_2 \leq 16$ (shipping and handling constraint) – Behaviour constraint.

$x_1 \leq 28$
 $x_2 \leq 14$ (Manufacturing constraint) – Behaviour constraint.

$x_1 \leq 14$
 $x_2 \leq 24$ (Sales Dept. limitations) – Behaviour constraint.

Finally, the design variables must be non-negative as $x_1, x_2 \geq 0$ – Side constraint.

[Here the formulation remains valid even when a design variable has zero value. The problem has two design variables and five inequality constraints. All functions of the problem are linear in variables x_1 and x_2 . Therefore, it is a linear programming problem.]

Graphical Solution:

$x_1 + x_2 \leq 16$ – g_1

x_1	0	16
x_2	16	0

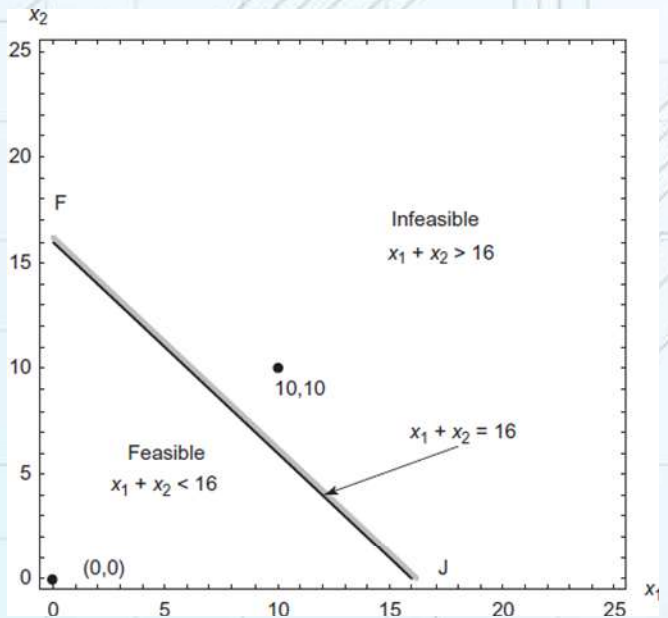
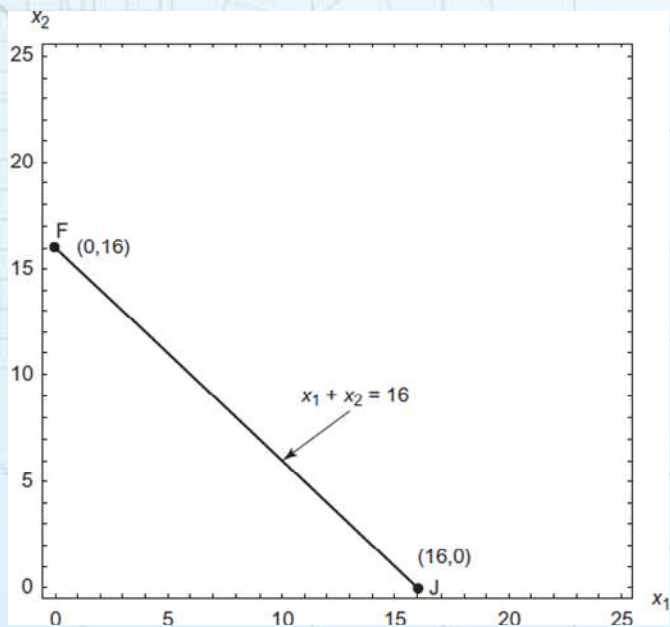


Fig. 6.10 Constraint boundary and Feasible/infeasible side for the inequality $x_1 + x_2 \leq 16$ in the profit maximization problem.

$$\frac{x_1}{1} + \frac{x_2}{14} = g_2$$

x_1	0	28	2	14
x_2	14	0	13	7

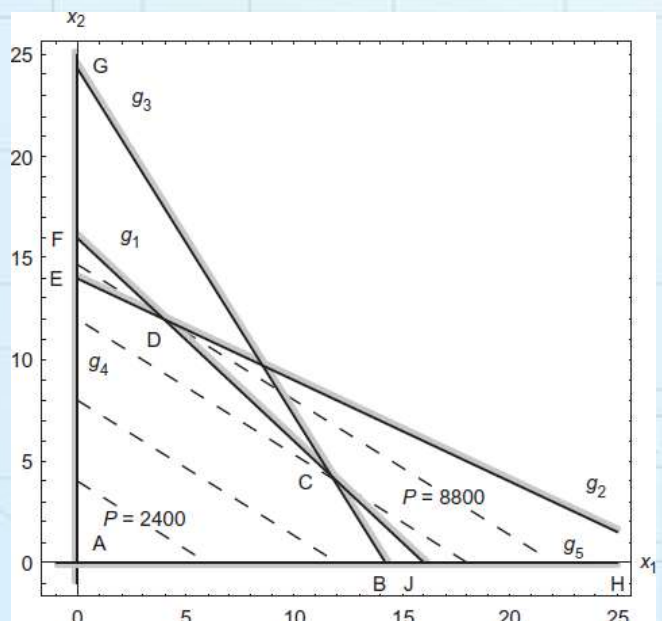
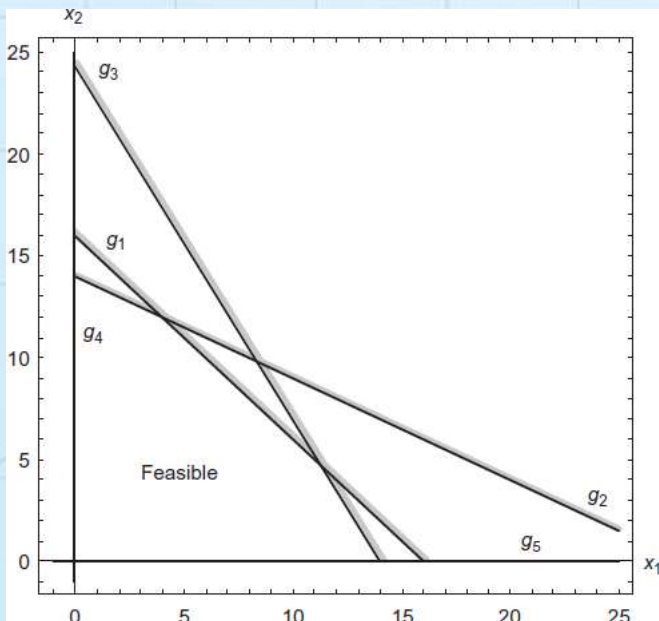


Fig. 6.11 Feasible region and Graphical solution to the profit maximization problem

$$\frac{x_1}{1} + \frac{x_2}{14} = g_3$$

x_1	0	14
x_2	24	0

$$x_1 \geq 0 \quad - g_4$$

$$x_2 \geq 0 \quad - g_5$$

Get the optimum function solution (To locate the optimum point)

$$f(x) = 400x_1 + 600x_2 \quad \text{Let at } (6,4)$$

$$= 400 \times 6 + 600 \times 4$$

$$= 4800$$

$$f(x) = 400x_1 + 600x_2 = 2400$$

$$f(x) = 400x_1 + 600x_2 = 7200$$

As the contours move up toward point D, feasible designs can be found with larger values for $f(x)$. It is clear from observation that point D has the largest value for $f(x)$ in the feasible region. Now simply read the coordinates of point D (4, 12) to obtain the optimum design, having a maximum value for the profit function as

$$f(x) = 400x_1 + 600x_2 = 8800.$$

Example 6.4: A manufacturing firm produces two machine parts using lathes, milling machines and grinding machines. The different machining times required for each part, the machining time available on different machines and the profit on each machine part are given in the following table.

Type of Machine	Machining Time required (in min)		Maximum time available per week (minutes)
	I (x)	II (y)	
Lathe	10	5	2500
Milling machine	4	10	2000
Grinding machine	1	1.5	450
Profit per unit	Rs. 50	Rs. 100	

Determine the number of parts I and II to be manufactured per week to maximize the profit.

Let A = number of parts I

B = number of parts II

1. Design vector :

The design vector is given by,

$$X = \begin{bmatrix} A \\ B \end{bmatrix} \dots\dots\dots(a)$$

2. Objective function :

The objective function to be maximized is the total profit. It is given by,

$$f(x) = 50 A + 100 B \dots\dots\dots (b)$$

3. Design constraints :

Machining time of lathe machine : $10A + 5B \leq 2500$ (c)

Machining time of milling machine : $4A + 10B \leq 2000$ (d)

Machining time of grinding machine : $A + 1.5B \leq 450$(e)

Non-negative constraints : $A \geq 0$ and integer(f)

$B \geq 0$ and integer(g)

From Equation (c),

$$\frac{10A + 5B}{2500} \leq 1$$

$$\frac{A}{250} + \frac{B}{500} \leq 1$$

..... (h)

From Equation (d),

$$\frac{4A + 10B}{2000} \leq 1$$

$$\frac{A}{500} + \frac{B}{200} \leq 1$$

..... (i)

From Equation (e),

$$\frac{A + 1.5B}{450} \leq 1$$

$$\frac{A}{450} + \frac{B}{300} \leq 1$$

.....(j)

4. Plotting of design constraints :

The above design constraints can be plotted in two dimensional design space (A-axis and B-axis), as shown in Fig. 6.12.

Design Constraint	Remark
$\frac{A}{250} + \frac{B}{500} \leq 1$	Straight line with A - intercept =250 and B - intercept = 500
$\frac{A}{500} + \frac{B}{200} \leq 1$	Straight line with A - intercept =500 and B - intercept = 200
$\frac{A}{450} + \frac{B}{300} \leq 1$	Straight line with A - intercept =450 and B - intercept = 300

1. Contours of objective function:

Coordinates of point L :

$$A = 0, B = 200$$

Coordinates of point N :

$$A = 250, B = 0$$

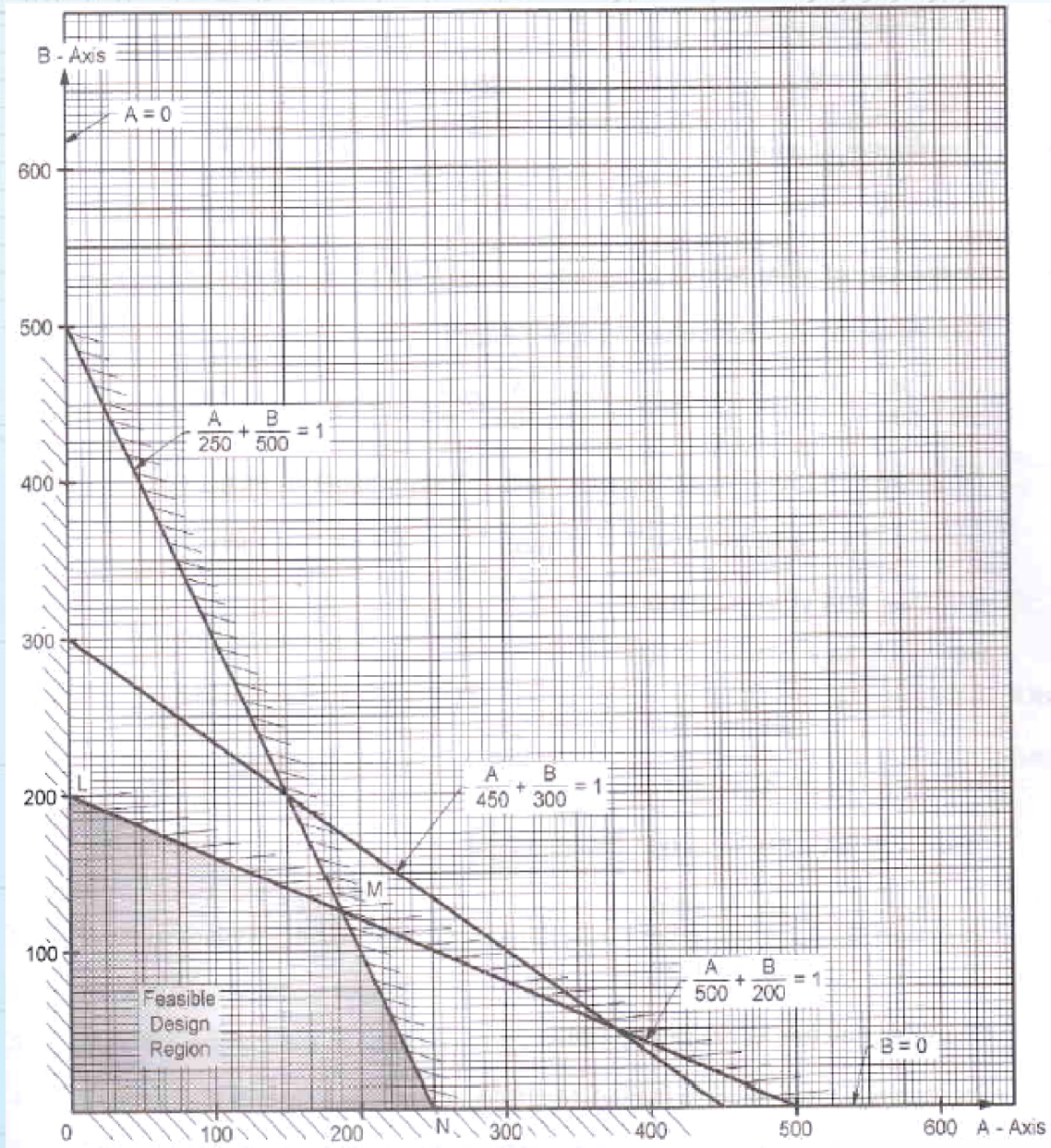


Fig.6.12 Graphical optimization

Coordinates of point M :

Point M can be obtained as intersection of lines,

$$\frac{A}{250} + \frac{B}{500} = 1$$

$$\frac{A}{500} + \frac{B}{200} = 1$$

$$500A + 250B = 125000 \quad \text{and} \quad 200A + 500B = 100000$$

$$5A + 2.5B = 1250 \quad \text{and} \quad 2A + 5B = 1000$$

$$10A + 5B = 2500$$

$$\underline{+ 2A + 5B = 1000}$$

$$8A = 1500$$

$$A = 187.5$$

$$B = 125$$

Objective function curve through M :

For objective function curve through M (187.5, 125),

$$\begin{aligned} f(X) &= P = 50A + 100B \\ &= 50 \times 187.5 + 100 \times 125 \\ &= 21875 \end{aligned}$$

Equation of objective function curve (line) through M is,

$$50A + 100B = 21875$$

$$\frac{50A}{21875} + \frac{100B}{21875} = 1$$
$$\frac{A}{437.5} + \frac{B}{218.75} = 1$$

Optimum solution:

As $P = f(X)$ is maximum at point M, the optimum solution is given by point M with,

$A = 187.5$;: 187 (As A has to be an integer)

and $B = 125$

$$\begin{aligned} P &= f(X) = 50A + 100B \\ &= 50 \times 187 + 100 \times 125 \\ &= 21850 \end{aligned}$$

Number of parts I, $A = 187$

Number of parts II, $B = 125$

Profit, $P = \text{Rs.}21850$

6.7 Optimum Design with Lagrange Multipliers

- The Lagrange Multiplier is a powerful method for finding optima in multivariable problems involving function constraints.

Consider the objective function

$$U = U_1(x, y, z)$$

Subject to functional constraints

$$T_1 = T_1(x, y, z) \text{ and}$$

$$T_2 = T_2(x, y, z)$$

Then, the new Lagrange Expression (LE) is

$$LE = U_1(x, y, z) + \lambda_1 \psi_1(x, y, z) + \lambda_2 \psi_2(x, y, z)$$

where λ_1 and λ_2 are the Lagrange multipliers.

The following conditions must then be satisfied at the optimum point :

$$\frac{\partial LE}{\partial x} = 0, \quad \frac{\partial LE}{\partial y} = 0, \quad \frac{\partial LE}{\partial z} = 0;$$

$$\frac{\partial LE}{\partial \lambda_1} = 0, \quad \frac{\partial LE}{\partial \lambda_2} = 0.$$

Example 6.5: A heat exchanger is to be designed for minimum cost. The design requirement indicated that 300 m of tube length in all be installed for achieving necessary heat transfer surface.

The total cost of installation has been approximated as:

- 1) Total cost of tubes = Rs.24500
- 2) Cost of shell = Rs.875D^{2.5}L
where D = shell diameter and
L = length of each tube ∴ Length of heat exchanger
- 3) Cost of floor space occupied by the heat exchanger = 700 D L.

The spacing of the tubes is such that 20 tubes can be accommodated in cross sectional area of 1 m² inside the shell. Determine the diameter D and length L of the heat exchanger to minimize costs.

The objective function is

$$c = 24500 + 875 D^{2.5} L + 700 D L$$

Subject to the following functional constraints :-

$$\ln 1 \text{ m}^2 \quad \text{---} \quad 20 \text{ tubes}$$

$$\ln \frac{\pi}{4} D^2 \text{ (total shell area)} \quad \text{---} \quad x \text{ (number of tubes each having length L)}$$

$$x = 5 \pi D^2$$

$$\begin{aligned} \text{Total running length of tubes} &= 300 \text{ m} \\ &= \text{Number of tubes} \times \text{length of each tube} \\ &= x L \end{aligned}$$

$$x = \frac{300}{L}$$

$$\frac{300}{L} = 5 \pi D^2$$

$$L = \frac{300}{5 \pi D^2}$$

$$\psi_1 = L - \frac{300}{5 \pi D^2}$$

Therefore, the Lagrange expression (L E) is

$$LE = 24500 + 875 D^{2.5} L + 700 DL + \lambda \left(L - \frac{60}{\pi D^2} \right)$$

$$\frac{\partial LE}{\partial D} = 2187.50 D^{1.5} L + 700 L + \lambda \left[\frac{0 + 60 \times 2 \pi D}{(\pi D^2)^2} \right] = 0$$

$$2187.50 D^{1.5} L + 700 L + 2 \lambda \left(\frac{60}{\pi D^3} \right) = 0 \quad \dots \text{ (a)}$$

$$\frac{\partial LE}{\partial L} = 875 D^{2.5} + 700 D + \lambda \quad \dots \text{ (b)}$$

$$\lambda = - 875 D^{2.5} - 700 D \quad \dots \text{ (c)}$$

$$\frac{\partial LE}{\partial \lambda} = L - \frac{60}{\pi D^2} = 0 \quad \dots \quad (d)$$

$$L = \frac{60}{\pi D^2} \quad \dots \quad (e)$$

Substituting (c) and (e) in (a)

$$2187.50 D^{1.5} \left(\frac{60}{\pi D^2} \right) + 700 \left(\frac{60}{\pi D^2} \right) + 2[-875 D^{2.5} - 700 D] \left(\frac{60}{\pi D^3} \right) = 0$$

$$\therefore 2187.50 D^{1.5} + 700 - 1750 D^{1.5} - 1400 = 0$$

$$\therefore 437.50 D^{1.5} = 700$$

$$\therefore D^{1.5} = 1.6$$

$$D = 1.368 \text{ m}$$

$$\text{Hence } L = \frac{60}{\pi D^2} = \frac{60}{\pi \times (1.368)^2} = 10.21 \text{ m}$$

Therefore, Cost

$$c = 24500 + 875 D^{2.5} L + 700 D L$$

$$= 24500 + 875 (1.368)^{2.5} \times 10.21 + 700 \times 1.368 \times 10.21$$

$$= 53,832 \text{ Rupees.}$$

Example 6.6: Find the solution using the Lagrange multiplier method:

$$\text{Minimize } f(x, y) = kx^{-1}y^{-2}$$

$$\text{Subject to } g(x, y) = x^2 + y^2 - a^2 = 0$$

The Lagrange function is

$$L(x, y, \lambda) = f(x, y) + \lambda g(x, y) = kx^{-1}y^{-2} + \lambda(x^2 + y^2 - a^2)$$

The necessary conditions for the minimum of $f(x, y)$ [Eqs. (2.38)] give

$$\frac{\partial L}{\partial x} = -kx^{-2}y^{-2} + 2x\lambda = 0 \quad (E_1)$$

$$\frac{\partial L}{\partial y} = -2kx^{-1}y^{-3} + 2y\lambda = 0 \quad (E_2)$$

$$\frac{\partial L}{\partial \lambda} = x^2 + y^2 - a^2 = 0 \quad (E_3)$$

Equations (E₁) and (E₂) yield

$$2\lambda = \frac{k}{x^3y^2} = \frac{2k}{xy^4}$$

from which the relation $x^* = (1/\sqrt{2})y^*$ can be obtained. This relation, along with Eq. (E₃), gives the optimum solution as

$$x^* = \frac{a}{\sqrt{3}} \quad \text{and} \quad y^* = \sqrt{2} \frac{a}{\sqrt{3}}$$

Example 6.7: Find the dimensions of a cylindrical tin (with top and bottom) made up of sheet metal to maximize its volume such that the total surface area is equal to $A_0 = 24\pi$.

If x_1 and x_2 denote the radius of the base and length of the tin, respectively, the problem can be stated as

$$\text{Maximize } f(x_1, x_2) = 7rx_1^2x_2$$

$$2\pi x_1^2 + 2\pi x_1x_2 = A_0 = 24\pi$$

The Lagrange function is

$$L(x_1, x_2, \lambda) = \pi x_1^2x_2 + \lambda(2\pi x_1^2 + 2\pi x_1x_2 - A_0)$$

and the necessary conditions for the maximum of f give

$$\frac{\partial L}{\partial x_1} = 2\pi x_1x_2 + 4\pi\lambda x_1 + 2\pi\lambda x_2 = 0 \quad (E_1)$$

$$\frac{\partial L}{\partial x_2} = \pi x_1^2 + 2\pi\lambda x_1 = 0 \quad (E_2)$$

$$\frac{\partial L}{\partial \lambda} = 2\pi x_1^2 + 2\pi x_1x_2 - A_0 = 0 \quad (E_3)$$

Equations (E₁) and (E₂) lead to

$$\lambda = -\frac{x_1x_2}{2x_1 + x_2} = -\frac{1}{2}x_1$$

$$x_1 = \frac{1}{2}x_2 \quad (E_4)$$

and Eqs. (E₃) and (E₄) give the desired solution as

$$x_1^* = \left(\frac{A_0}{6\pi}\right)^{1/2}, \quad x_2^* = \left(\frac{2A_0}{3\pi}\right)^{1/2}, \quad \text{and } \lambda^* = -\left(\frac{A_0}{24\pi}\right)^{1/2}$$

This gives the maximum value of f as

$$f^* = \left(\frac{A_0^3}{54\pi}\right)^{1/2}$$

If $A_0 = 24\pi$, the optimum solution becomes

$$x_1^* = 2, \quad x_2^* = 4, \quad \lambda^* = -1, \quad \text{and } f^* = 16\pi$$

6.8 Johnson's Method of Optimum Design

In Johnson's method of Optimum Design, any mechanical element design can have 3 forms of equations:-

(a) Primary Design Equation (PDE)

This is the most important design equation, which expresses the most significant functional requirement or the most significant undesirable effect. For example, the primary design

equation for an airplane shaft may be the one which expresses its weight, which is an undesirable effect and needs to be minimized.

(b) Subsidiary Design Equations (SDE)

In a mechanical element, design equations other than PDE are called Subsidiary Design Equations (SDE). SDE express either functional requirements or significant undesirable effects, whether they are directly specified or indirectly implied. The most important SDE are the stress equations, which are generally implied.

(c) Limit Equations or Constraints (LE)

As mentioned previously, various parameters have certain ranges which are expressed in mathematically simple limit equations. For example, the limitations in stress are imposed by the strength of the material. Limitations on geometry are imposed by certain functional requirement or space constraints.

Basic steps in method of Optimum Design:-

1) Problem Formulation:

- a. Decide on basic configuration in the form of sketches identifying loads, deflections and stresses.
- b. Summarising the significant constraints, compile the Primary Design Equation, Subsidiary Design Equation and the Limit Equation.
- c. The equations are so combined that the total number of equations is equal to the number of free variables.
- d. These equations are then studied for optimum design.

The above procedure would be clarified in the following sections. Three types of problems are encountered in the method of Optimum Design:-

- I. Case of Normal Specifications (NS)
- II. Case of Redundant Specifications (RS)
- III. Case of Incompatible Specifications (IS)

I. Case of Normal Specifications

This is the case in which the final formulation consists of a single PDE. For this type of problem, $n_f \geq N_S$, where n_f is the number of free variables and N_S is the number of SDEs. A free variable is one with no constraints imposed by the Limit Equations.

$$n_v = n_f + n_c$$

where

n_v = Total number of Variables and

n_c = Total number of constrained variables.

In this case, it is possible to uniquely determine the optimum solution.

Example 6.8: Design a tensile bar of length $L = 200$ mm to carry a tensile load of 5 kN for minimum cost, out of the following materials:

Material	Mass Density (kg/m ³)	Material cost (Rs / N Weight)	Yield strength (MPa)
Steel	7500	16	130
AL-alloy	3000	32	50
Titanium Alloy	4800	480	90
Magnesium Alloy	2100	32	20

Fig. 6.13 Simple Tensile Bar

(1) PDE (cost of material is the criterion of optimization)

Let c_m = Cost of the bar in Rs.

P = Mass density in kg/m³.

A = Area in m²

L = Length in m

c = Cost of material / weight in Rs/N

c_m = Weight of the bar (N) x c

= (Mass Density x Volume) x c

= $p ALgc$

PDE is — $c_m = p ALgc$

(2) SDE

$$c_r = P/A$$

where P is the specified load

(3) Linear Equation (LE)

$$c_r : S \text{ cr}_y / F.S$$

(4) It is observed that c , p , c_r are material parameters, A is the geometrical parameter, c_r and c_m are undesirable effects, P , L and $F.S$ are functional requirements.

We observe that there is one free variable (A) and one SDE

$$n_f = N_s$$

Simplifying and combining equations

$$\begin{aligned}
 c_m &= c \rho A L g \quad \text{and} \\
 A &= \frac{P}{\sigma} \quad \text{we get} \\
 c_m &= c \rho \frac{P}{\sigma} L g \\
 \sigma &= \frac{S_y}{\text{fos}} \\
 c_m &= \left(\frac{c \rho P L g}{S_y} \right) \text{fos} \\
 &= \left(\frac{c \rho}{S_y} \right) g P L \text{fos} \quad \dots
 \end{aligned}$$

Here, P, L, F.S. are specified quantities and cannot be changed. (g is a constant).
 Therefore, from the equation, the material selection factor (MSF) is $c \rho / c_y$
 The cost would be minimum for the material having lowest value of (MSF)

Material	$\frac{MSF}{c} = \frac{S_y}{c \rho}$
Steel	$\frac{16 \times 7500}{130} = 923.077$
AL-alloy	$\frac{32 \times 3000}{50} = 1920$
Titanium Alloy	$\frac{480 \times 4800}{90} = 25600$
Magnesium Alloy	$\frac{32 \times 2100}{20} = 3360$

It is seen that steel has the lowest MSF and hence should be used to cost for achieve optimization in cost for given conditions

Assuming F.S. = 2, the free variable A (area) can be calculated by using SDE i.e.

$$\begin{aligned}
 A &= \frac{P}{\sigma} = \frac{P}{S_y} \text{fos} \\
 &= \frac{5 \times 10^3 \times 2}{130} = 76.923 \text{ mm}^2
 \end{aligned}$$

Example 6.9: In a light weight equipment, a shaft is transmitting a torque of 900 Nm and is to have a rigidity of 90 Nm/degree. Assume a factor of safety of 1.5 based on yield stress. Design the shaft with minimum weight. What will be the change in design for minimum cost? Assume maximum shear stress theory of failure. Use the following data for the materials:-

Material	Mass Density (kg/m ³)	Material cost (Rs / N Weight)	Yield strength (MPa)	Shear Modulus (GPa)
Steel	8500	16	130	80
AL-alloy	3000	32	50	26.7
Titanium Alloy	4800	480	90	40
Magnesium Alloy	2100	32	20	16

Let p = Mass Density in kg/m³

K = Torsional rigidity in Nm/radian

M_T = Torque rating in Nm

F.S. = 1.5

G = Modulus of rigidity in N/m²

S_y = Yield stress in N/m²

d = Diameter of shaft in m,

L = Length of shaft in m

W = Weight of shaft in N

c_m = Material cost/weight (Rs/N)

(1) PDE:

$$W = p \times L \times g \times d^2 \times n/4$$

4) Combining Equations,

$$W = \rho \frac{\pi}{4} d^2 \left(\frac{\pi d^4 G}{32 K} \right) g$$

$$= \frac{\pi^2}{128} \frac{\rho G}{K} d^6 g$$

$$= \frac{\pi^2}{128} \left(\frac{\rho G}{K} \right) \left(\frac{16 M_T}{\pi \tau_{\max}} \right)^2 g$$

$$= 2 g \left(\frac{\rho G}{K} \right) \left(\frac{M_T}{\tau_{\max}} \right)^2$$

$$= 2 g \left(\frac{M_T}{K} \right)^2 (\rho G) \left(\frac{1}{\tau_{\max}} \right)^2$$

W will be the minimum when $\tau = \tau_{\max}$

$$\tau_{\max} = \frac{0.5 S_y}{\text{fos}}$$

$$W = 2 \left(\frac{M_T}{K} \right)^2 (\rho G) \frac{(\text{fos})^2}{0.25 S_y^2}$$

$$= 8 g \left(\frac{M_T \text{ fos}}{K} \right)^2 \left(\frac{\rho G}{S_y^2} \right)$$

The MSF = $\frac{\rho G}{S_y^2}$

M_T , F.S. and K are specified functional requirements. Hence, the minimum MSF will mean minimum weight.

Calculating from the data given

Material	For Weight $MSF \propto \frac{\rho G}{S_y^2}$	For Cost $MSF \propto \frac{c \rho G}{S_y^2}$
Steel	$\frac{8500 \times 80 \times 10^9}{(130 \times 10^6)^2} \propto 0.04024$	0.6438
AL-alloy	$\frac{3000 \times 26.7 \times 10^9}{(50 \times 10^6)^2} \propto 0.03204$	1.0253

Titanium Alloy	$4800 \square 40 \square 10^9$ $(90 \square 10^6)^2$ $\square 0.02370$	11.3760
Magnesium Alloy	$2100 \square 16 \square 10^9$ $(20 \square 10^6)^2$ $\square 0.08400$	2.6880

Minimum MSF (0.02370) is for Titanium Alloy and hence it should be used to achieve, minimum weight for given conditions.

$$d = \left(\frac{16 M_T}{\pi \tau_{\max}} \right)^{1/3}$$

$$\tau_{\max} = \frac{0.5 S_y}{\text{fos}} = \frac{0.5 \times 90}{1.5}$$

$$= 30 \text{ N/mm}^2$$

$$d_{\text{titanium}} = \left(\frac{16 \times 900 \times 10^3}{\pi \times 30} \right)^{1/3}$$

$$= 53.46$$

$$L = \frac{\pi d^4 G}{32 K}$$

$$K = 90 \text{ N m/degree}$$

$$= \frac{90 \times 180}{\pi} \times 10^3$$

$$= 5156.6202 \times 10^3 \text{ N mm/rad}$$

$$L_{\text{titanium}} = \frac{\pi \times 53.46^4 \times 40 \times 10^3}{32 \times 5156.6202 \times 10^3}$$

$$= 6220.289 \text{ mm}$$

b) If the criterion for optimisation is cost, then,

$$c_m = cW$$

$$\therefore c_m = c \left[8 g \left(\frac{M_T \text{ fos}}{K} \right)^2 \left(\frac{\rho G}{S_y^2} \right) \right]$$

$$= 8 g \left(\frac{M_T \text{ fos}}{K} \right)^2 \left(\frac{c \rho G}{S_y^2} \right)$$

$$\text{The MSF} = \left(\frac{c \rho G}{S_y^2} \right)$$

From the table, it is seen that the minimum MSF (0.6438) is for steel alloy which will result, in minimum cost.

$$\begin{aligned}\tau_{\max} &= \frac{0.5 \times 130}{15} \\ &= 43.33 \text{ N/mm}^2 \\ d_{\text{steel}} &= \left(\frac{16 \times 900 \times 10^3}{\pi \times 43.33} \right)^{1/3} \\ &= 47.293 \text{ mm} \\ L_{\text{steel}} &= \frac{\pi \times 47.293^4 \times 80 \times 10^3}{32 \times 5156.6202 \times 10^3} \\ &= 7619.25 \text{ mm}\end{aligned}$$

II. Case of Redundant Specifications:

This is a case in which the final formulation consists of a system of two or more equations. For the case of redundant specifications, it is not possible to include the effects of all SDEs and all LEs in a single PDE. This case is generally characterized by the necessity of eliminating a limited parameter in order to combine a SDE with PDE. Hence, there is generally an excess number of LEs for such cases and a special procedure is to be followed for problem solving.

The case of redundant specification occurs when

$$n_f < N_s$$

III. Case of Incompatible Specifications:

The case of incompatible specifications is a special case of redundant specifications where the SDE and LEs are such that it is impossible to satisfy them all with any one physically feasible design of the mechanical element. In such cases, no optimum design solution exists.

Example 6.10 : A simple tensile bar is subjected to the specified constant tensile force 'F'. Design the bar with the objective of minimizing the material cost using factor of safety 'N_f'. The following limitation is specified in the optimum design.

$$L_{\min} \leq L \leq L_{\max}$$

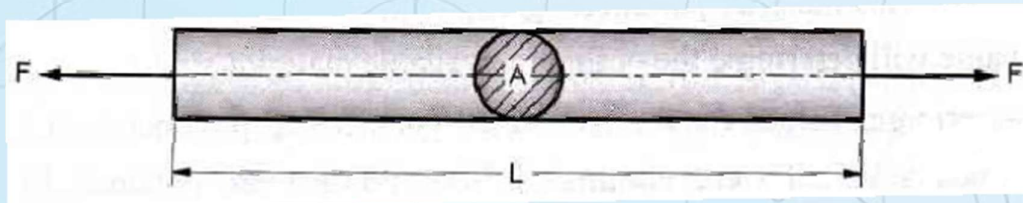


Fig. 6.14

C_m = material cost of the tensile bar, Rs.

c = cost per unit mass for the bar material, Rs /kg

p = mass density of the bar material, kg / m³

A = cross-sectional area of the bar, m²

L = length of the bar, m
 F = tensile force acting on the bar, N
 N_f = factor of safety
 σ_t = tensile stress induced in the bar, N / m²
 S_{yt} = yield strength of the bar material, N / m²

1. Primary design equation (P.D.E.):

- The most significant undesirable effect to be minimized is the material cost of the bar and is given by,

$$C_m = \text{Cost per unit mass of the bar} \times \text{Mass of the bar}$$

$$C_m = c \times p \times A \times L$$

$$\text{or } C_m = c \times p \times A \times L, \text{ Rs} \dots\dots\dots(a)$$

2. Subsidiary design equation (S.D.E.) :

- The tensile stress induced in the bar is given by,

$$\sigma_t = F/A \dots\dots\dots(b)$$

3. Limit equations (L.E.):

- The limit equations are,

$$\sigma_t : S_{yt} / N_f \dots\dots\dots(c)$$

$$L_{min} : S \quad L : S \quad L_{max} \dots\dots\dots(d)$$

4. Classification of parameters :

	Specified	Limited	Unspecified and Unlimited
Functional Requirement Parameters	F, N_f		
Undesirable Effect Parameters		σ_t	C_m
Geometrical Parameters		L	A
Material Parameters		c, p, S_{yt}	

5. Combining S.D.E. with P.D.E. :

- S.D.E. [Equation (b)] is combined with P.D.E. [Equation (a)] by eliminating the unspecified and unlimited parameter A. Therefore,

$$C_m = c \times p \times A \times L / \sigma_t \dots\dots\dots(e)$$

- This is developed P.D.E. From Equation (e) it is seen that, the effects of both the limit equations i.e. Equations (c) and (d) can be included in the developed P.D.E.
- Therefore, this is the base of normal specifications.

6. Combining limit equations with P.D.E. :

- The next step is to include the effects of all limit equations in the developed P.D.E.
- $C_m \propto 1 / \sigma_t$, hence for minimizing ' C_m ', the parameter ' σ_t ' should be placed at its upper limit i.e. S_{yt} / N_f .
- Again, $C_m \propto L$, and hence for minimizing ' C_m ' the parameter 'L' should be placed at its lower limit i.e. L_{min} .

- Substituting these limiting values, Equation (e) can be written as,

$$C_m = \frac{c \rho F}{S_{yt} / N_f} \cdot L_{min}$$

$$= \frac{c \rho F N_f L_{min}}{S_{yt}}$$

or $C_m = [F N_f] [L_{min}] [c \rho / S_{yt}] \dots\dots\dots(f)$

- This is final P.D.E. In order to minimize 'C_m', the material selection factor [c ρ / S_{yt}] should be minimum.

7. Selection of material :

- The material selection factor [c ρ / S_{yt}] is calculated for all the available material and the material with the lowest value of [c ρ / S_{yt}] is selected.

8. Determination of eliminated parameter :

- The eliminated parameter 'A' can be determined by using S.D.E. [Equation (b)]. Therefore, substituting the limiting value of cr_t Equation (b),

$$\frac{S_{yt}}{N_f} = \frac{F}{A}$$

$$\therefore A = \frac{F \cdot N_f}{S_{yt}}$$

9. Determination of optimum quantity :

- Finally the most significant undesirable effect to be minimized i.e. 'C_m', is determined by using final P.D E [Equation (f)].

Example 6.11: A tensile bar of circular cross-section is subjected to the cyclic tensile force which varies from zero to maximum. Design the bar with the objective of maximizing the energy absorption capacity using the factor of safety 'N_f'. The following limitations are to be used in the design due to space and assembly restrictions.

d :S d_{max}

L_{min} :S L :S L_{max}

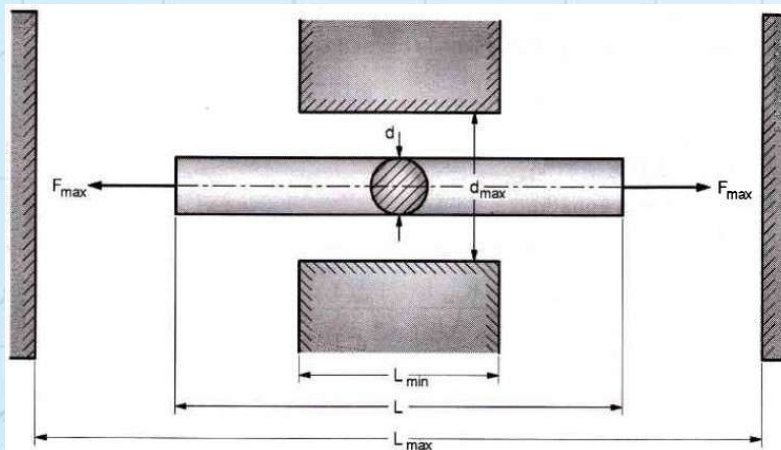


Fig. 6.15

The endurance limit of the test specimen is given by,

$$S_e = 0.5 S_{ut}$$

The endurance limit of the tensile bar is given by,

$$S_e' = K_a \cdot K_b \cdot K_c \cdot K_d \cdot K_e \cdot K_g \cdot S_e'$$

$$S_e' = K_a \cdot K_b \cdot K_c \cdot K_d \cdot K_e \cdot K_g \cdot 0.5 S_{ut}$$

$$S_e = K S_{ut} \dots\dots\dots(a)$$

Where $K = K_a \cdot K_b \cdot K_c \cdot K_d \cdot K_e \cdot K_g \times 0.5$

U = energy absorption capacity of the tensile bar, N-mm

L = length of the bar, mm

A = cross-sectional area of the bar, mm^2

d = diameter of the bar, mm

F_{max} = maximum force acting on the bar during the cycle, N

δ_{max} = maximum elongation of the bar during the cycle, N

σ_{rmax} = maximum stress induced in the bar, N/mm^2

σ_m = mean stress, N/mm^2

σ_a = stress amplitude, N/mm^2

N_f = factor of safety

S_{yt} = yield strength of the bar material, N/mm^2

S_{ut} = ultimate tensile strength of the bar material, N/mm^2

S_e = endurance limit of the bar, N/mm^2

E = modulus of elasticity of the bar material. N/mm^2

1. Primary design equation (P.D.E.) :

The most significant functional requirement to be maximized is the energy absorption capacity of the tensile bar and is given by,

$$U \propto \frac{1}{2} F_{max} \delta_{max} \dots\dots\dots(b)$$

2. Subsidiary design equations (S.D.E.):

The maximum elongation of the tensile bar is given by,

$$\delta_{max} = \frac{F_{max} L}{A E} = \frac{F_{max} L}{\frac{\pi}{4} d^2 E}$$

or $\delta_{max} = \frac{4 F_{max} L}{\pi d^2 E} \dots\dots\dots(c)$

The maximum tensile stress induced in the bar is given by,

$$\sigma_{\max} = \frac{F_{\max}}{A} = \frac{F_{\max}}{\frac{\pi}{4} d^2}$$

$$\sigma_{\max} = \frac{4 F_{\max}}{\pi d^2}$$

.....(d)

3. Limit equations (L.E.) :

Fig. 6.16 Repeated stress

According to the Soderberg criterion, for the safety of the bar against the fatigue failure,

$$\frac{\sigma_m}{S_{yt}} + \frac{\sigma_a}{S_e} \leq \frac{1}{N_f}$$

For the repeated stress, as shown in Fig. 6.16, the mean stress and the stress amplitude are given by,

$$\sigma_m = \frac{\sigma_{\max} + \sigma_{\min}}{2} = \frac{\sigma_{\max} + 0}{2} = \frac{\sigma_{\max}}{2} \quad \dots(f)$$

$$\sigma_a = \frac{\sigma_{\max} - \sigma_{\min}}{2} = \frac{\sigma_{\max} - 0}{2} = \frac{\sigma_{\max}}{2} \quad \dots(g)$$

Substituting Equations (a), (f) and (g) in Equation (e), we get,

$$\frac{\sigma_{\max}/2}{S_{yt}} + \frac{\sigma_{\max}/2}{K \cdot S_{ut}} \leq \frac{1}{N_f}$$

$$\frac{\sigma_{\max}}{2} \left(\frac{1}{S_{yt}} + \frac{1}{K S_{ut}} \right) \leq \frac{1}{N_f}$$

$$\frac{\sigma_{\max}}{2} \left(\frac{K S_{ut} + S_{yt}}{K S_{ut} \cdot S_{yt}} \right) \leq \frac{1}{N_f}$$

$$\text{or } \sigma_{\max} \leq \frac{2}{N_f} \left(\frac{K S_{ut} \cdot S_{yt}}{K S_{ut} + S_{yt}} \right) \quad \dots(h)$$

$$d \leq d_{\max} \quad \dots(i)$$

$$L_{\min} \leq L \leq L_{\max} \quad \dots(j)$$

The Equations (h), (i) and (j) are the limit equations.

4. Classification of parameters :

	Specified	Limited	Unspecified and Unlimited
Functional Requirement Parameters	N_f		U, F_{max}, σ_{max}
Undesirable Effect Parameters		$c r_{max}$	
Geometrical Parameters		L, d	A
Material Parameters		c, p, S_{yt}	

5. Combining S.D.E.'s with P.D.E. :

S.D.E. (c) is combined with P.D.E. [Equation (b)] by eliminating the unspecified and unlimited parameter ' σ_{max} '. Therefore,

$$U = \frac{1}{2} F_{max} \times \frac{r_{max}}{\pi d^2 E}$$

$$\text{or } U = \frac{2 (F_{max})^2 L}{\pi d^2 E} \dots\dots\dots(k)$$

Again S.D.E. (d) is combined with P.D.E. [Equation (k)] by eliminating the unspecified and unlimited parameter F_{max} . Hence,

$$U = \frac{2}{\pi d^2 E} \left(\frac{r_{max}}{4} \right)^2 L$$

$$\text{or } U = \frac{\pi \sigma_{max}^2 d^2 L}{8 E} \dots\dots\dots(l)$$

This is developed P.D.E. From Equation (l) it is seen that, the effects of all limit equations i.e. Equations (h), (i) and (r) can be included in the developed P.D.E.

Therefore, this is the case of normal specifications.

6. Combining limit equations with P.D.E. :

The next step is to include the effects of all limit equations in the developed P.D.E.

$U \propto c r_{max}^2$, hence for maximizing 'U', the parameter $c r_{max}$, should be placed at its upper limit

$$\text{i.e. } \frac{2}{\pi} \left[\frac{K S_{ut} \cdot S_{yt}}{K S_{ut} + S_{yt}} \right]^2$$

$U \propto d^2$, hence for maximizing 'U', the parameter d, should be placed at its upper limit i.e. d_{max} .

Again $U \propto L$, hence for maximizing 'U', the L, should be placed at its upper limit i.e. L_{max} .

Substituting these limiting values, Equation (l) can be written as,

$$U = \frac{\pi}{8 E} \left\{ \left[\frac{2}{N_f} \left(\frac{K S_{ut} \cdot S_{yt}}{K S_{ut} + S_{yt}} \right) \right]^2 \right\} d_{max}^2 \cdot L_{max}$$

$$\text{or } U = \frac{\pi}{2} \left[\frac{1}{N_f^2} \right] \left[d_{max}^2 \cdot L_{max} \right] \left[\frac{1}{E} \left(\frac{K S_{ut} \cdot S_{yt}}{K S_{ut} + S_{yt}} \right)^2 \right] \dots\dots\dots(m)$$

This is final P.D.E. In order to maximize 'U', the material selection factor

$$\frac{1}{E} \left[\frac{K S_{ut} \cdot S_{yt}}{K S_{ut} + S_{yt}} \right]^2$$

should be maximum.

7. Selection of material:

The material selection factor
$$\frac{1}{L} \left(\frac{K \cdot S_{ut} \cdot S_{yt}}{E \cdot \left(\frac{K \cdot S_{ut} \cdot S_{yt}}{E} \right)^{1/2}} \right)^2$$
 is calculated for all the available materials

and the material with the largest value of
$$\frac{1}{L} \left(\frac{K \cdot S_{ut} \cdot S_{yt}}{E \cdot \left(\frac{K \cdot S_{ut} \cdot S_{yt}}{E} \right)^{1/2}} \right)^2$$
 is selected.

8. Determination of eliminated parameters :

The eliminated parameters F_{max} and d_{max} are determined by using S.D.E.'s i.e. Equations (c) and (d).

Substituting limiting values of d_{max} and 'd' in Equation (d),

$$\frac{2}{N_f} \left(\frac{K S_{ut} \cdot S_{yt}}{K S_{ut} + S_{yt}} \right) = \frac{4 F_{max}}{\pi d_{max}^2}$$

$$\therefore F_{max} = \frac{\pi d_{max}^2}{2 N_f} \left(\frac{K S_{ut} \cdot S_{yt}}{K S_{ut} + S_{yt}} \right)$$

Substituting limiting values of 'L' and 'd' in Equation (c),

$$\delta_{max} = \frac{4 F_{max} L_{max}}{\pi d_{max}^2 E}$$

9. Determination of optimum quantity :

Finally the most significant functional requirement to be maximized i.e. 'U' is determined by using the final P.D.E. [Equation (m)].

References :

- Engineering Optimization – S. S. Rao
- Introduction to Optimum Design – J. S. Arora
- Machine & CAD – Farzak Haideri