GUJARAT TECHNOLOGICAL UNIVERSITY

BE- 1 SEMESTER– PAPER SOLUTION – WINTER 2024

Subject Name & Code:

Mathematics-I BE01000041

Q-1: (a) Find $\lim_{x\to 0}\left(\frac{1}{x^2}-\frac{1}{sin^2x}\right)$ **(3 Marks)**

Answer:

We simplify the given expression:

$$\lim_{x\to 0}\left(\frac{1}{x^2}-\frac{1}{\sin^2 x}\right)=\lim_{x\to 0}\left(\frac{\sin^2 x - x^2}{x^2\sin^2 x}\right)$$

Using the standard expansion:

$$\sin x = x-\frac{x^3}{6}+\frac{x^5}{120}-\cdots \Rightarrow \sin^2 x = x^2-\frac{x^4}{3}+\dots$$

So,

$$\sin^2 x - x^2 = -\frac{x^4}{3}+\dots$$

Substitute into the limit:

$$\lim_{x\to 0}\frac{-\frac{x^4}{3}}{x^2(x^2-\frac{x^4}{3})}=\lim_{x\to 0}\frac{-\frac{x^4}{3}}{x^4\left(1-\frac{x^2}{3}\right)}$$

$$=\lim_{x\to 0}\frac{-1}{3\left(1-\frac{x^2}{3}\right)}=\frac{-1}{3}$$

✅ **Final Answer:**

$$\boxed{-\frac{1}{3}}$$

Q-1: (b)

Write second derivative test for local extrema and find the extrema of
$f(x) = x^4 - 4x^3 + 10$ **(4 Marks)**

Answer:

Part 1: Second Derivative Test (Theory)

Let f(x) be a twice-differentiable function. To find local extrema:

1. **Find critical points**: Solve f'(x)=0.
2. **Apply second derivative**:
   - If f''(x)>0, local **minimum** at that point.
   - If f''(x)<0, local **maximum**.
   - If f''(x)=0, the test is **inconclusive**.

Part 2: Apply to Given Function

Given:

$$f(x) = x^4 - 4x^3 + 10$$

First derivative:

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$$

Set f'(x)=0f to find critical points:

$$4x^2(x-3) = 0 \Rightarrow x = 0,\ 3$$

Second derivative:

$$f''(x) = 12x^2 - 24x = 12x(x-2)$$

Now evaluate:

- At x=0:

$$f''(0) = 12(0)(0-2) = 0 \Rightarrow \text{Test inconclusive}$$

- At x=3:

$$f''(3) = 12(3)(1) = 36 > 0 \Rightarrow \text{Local Minimum at } x = 3$$

Let's analyze x=0 using first derivative sign change:

- For $x < 0$, say $x = -1$:
  $f'(x) = 4(-1)^2(-1-3) = 4(1)(-4) = -16$
- For $x > 0$, say $x = 1$:
  $f'(x) = 4(1)^2(1-3) = 4(1)(-2) = -8$

Since sign of f'(x) does not change around x=0, **no extremum at x=0**.

Final Result:

- **Local minimum** at x=3
- $f(3) = 3^4 - 4(3)^3 + 10 = 81 - 108 + 10 = -17$

✅ **Local minimum value is** −17 at x=3

Q-1: (c)

Expand e^x into the power series of x. Find number of terms required to calculate e with an error less than 10^{-4}.

(7 Marks)

Answer:

Part 1: Power Series Expansion

The exponential function e^x has the following Taylor (Maclaurin) series expansion:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

To find the value of e = e^1, we substitute x=1:

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots$$

We want to approximate e such that the **error** is less than 10^{-4}.

Part 2: Error Estimation

In a Taylor series, the **remainder (error)** after n terms is:

$$R_{n+1} = \frac{x^{n+1}}{(n+1)!}$$

For x=1, we want:

$$\frac{1}{(n+1)!} < 10^{-4}$$

Now check values:

- For $n+1=6$, $\frac{1}{6!} = \frac{1}{720} \approx 1.39 \times 10^{-3}$ ❌
- For $n+1=7$, $\frac{1}{7!} = \frac{1}{5040} \approx 1.98 \times 10^{-4}$ ❌
- For $n+1=8$, $\frac{1}{8!} = \frac{1}{40320} \approx 2.48 \times 10^{-5}$ ✅

So, we need terms up to 1/7!, i.e., **8 terms**.

Final Answer:

- Power series of e^x:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

- To compute e = e^1 with error less than 10^{-4}, use first **8 terms**:

$$\boxed{e \approx 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{7!}}$$

✅ **Number of terms required = 8**

Q-2: (a)

Define Beta and Gamma functions and evaluate $\int_0^\infty \frac{x^3(1+x^2)}{(1+x)^{10}}dx$ **(3 Marks)**

Answer:

Part 1: Definitions

✅ **Gamma Function:**

$$\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}dx, \quad \text{for } n > 0$$

✅ **Beta Function:**

$$B(m,n)=\int_0^1 t^{m-1}(1-t)^{n-1}dt=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)},\quad \text{for } m,n>0$$

Part 2: Evaluate the Integral

Given:

$$I=\int_0^\infty \frac{x^3(1+x^2)}{(1+x)^{10}}dx=\int_0^\infty \frac{x^3+x^5}{(1+x)^{10}}dx=\int_0^\infty \frac{x^3}{(1+x)^{10}}dx+\int_0^\infty \frac{x^5}{(1+x)^{10}}dx$$

Let’s evaluate each term using Beta function transformation.

Substitute:

Let $x=\frac{t}{1-t}\Rightarrow t=\frac{x}{1+x}\Rightarrow dx=\frac{dt}{(1-t)^2}$

Using this substitution, known result:

$$\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx=B(m,n)$$

Now,

- For $\int \frac{x^3}{(1+x)^{10}}dx$:
  This is $\int_0^\infty \frac{x^3}{(1+x)^{10}}dx=B(4,6)$
- For $\int \frac{x^5}{(1+x)^{10}}dx$:
  This is $\int_0^\infty \frac{x^5}{(1+x)^{10}}dx=B(6,4)$

Use identity:

$$B(m,n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

So,

$$B(4,6)=\frac{\Gamma(4)\Gamma(6)}{\Gamma(10)}=\frac{3!\cdot 5!}{9!}=\frac{6\cdot 120}{362880}=\frac{720}{362880}=\frac{1}{504}$$

$$B(6,4)=B(4,6)=\frac{1}{504}$$

Hence, the total value of integral:

$$I = B(4,6) + B(6,4) = \frac{1}{504} + \frac{1}{504} = \boxed{\frac{1}{252}}$$

✅ **Final Answer:**

$$\int_0^\infty \frac{x^3(1+x^2)}{(1+x)^{10}} dx = \boxed{\frac{1}{252}}$$

Q-2: (b) Discuss the convergence of $\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx$ **(4 Marks)**

Answer:

This is an **improper integral** over an infinite interval. We split it into two parts and check for convergence:

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$$

Step 1: Evaluate

We use the standard integral formula:

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$$

So,

- For the first part:

$$\int_0^\infty \frac{1}{1+x^2}\,dx = \lim_{t\to\infty} \int_0^t \frac{1}{1+x^2} dx = \lim_{t\to\infty} [\tan^{-1} x]_0^t = \frac{\pi}{2}$$

For the second part:

$$\int_{-\infty}^0 \frac{1}{1+x^2}\,dx = \lim_{t\to-\infty} \int_t^0 \frac{1}{1+x^2} dx = \lim_{t\to-\infty} [\tan^{-1} x]_t^0 = \frac{\pi}{2}$$

Step 2: Add Both Parts

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \frac{\pi}{2} + \frac{\pi}{2} = \boxed{\pi}$$

✅ **Final Answer:**

- The integral **converges**, and
- Its value is π

Q-2: (c)

The circle $x^2 + y^2 = a^2$ is rotated about x −axis to generate a sphere, find its volume and sketch the region. **(7 Marks)**

Answer:

Step 1: Understanding the Geometry

The equation:

$$x^2 + y^2 = a^2$$

represents a **circle of radius a** centered at the origin.

When this curve is revolved about the x-axis, the region it bounds generates a **sphere of radius aaa**.

Step 2: Volume by Disk Method

To find the volume generated by rotating about the x-axis, we use the **disk method**:

$$V = \pi \int_{-a}^{a} [f(x)]^2 dx$$

From the circle equation:

$$y = \sqrt{a^2 - x^2} \Rightarrow f(x) = \sqrt{a^2 - x^2} \Rightarrow [f(x)]^2 = a^2 - x^2$$

Now apply the volume formula:

$$V = \pi \int_{-a}^{a} (a^2 - x^2) dx$$

Since the integrand is **even**, simplify using symmetry:

$$V = 2\pi \int_0^a (a^2 - x^2)dx$$

Step 3: Evaluate the Integral

$$\int_0^a (a^2 - x^2)dx = \left[a^2x - \frac{x^3}{3}\right]_0^a = a^3 - \frac{a^3}{3} = \frac{2a^3}{3}$$

Now multiply by 2π:

$$V = 2\pi \cdot \frac{2a^3}{3} = \boxed{\frac{4}{3}\pi a^3}$$

✅ **Final Answer:**

- Volume of the sphere generated is:

$$\boxed{\frac{4}{3}\pi a^3}$$

OR

Q-2: (c)

Find the surface area of a solid generated by the revolution of the circle $r = 2acos\theta$ about the initial line. **(7 Marks)**

Answer:

Step 1: Understand the Geometry

The given polar equation:

r=2acosθ

represents a **circle of radius a** with center at (a,0)) in Cartesian coordinates.

We're asked to **rotate it about the x-axis** to get a surface of revolution and compute its **surface area**.

Step 2: Formula for Surface Area in Polar Coordinates

The surface area S of a curve given by r=f(θ), when revolved about the **x-axis (initial line)**, is:

$$S = \int_{\theta_1}^{\theta_2} 2\pi r \sin\theta \cdot \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Step 3: Apply the Formula

Given r=2acosθ

- Derivative:

$$\frac{dr}{d\theta} = -2a\sin\theta$$

- Now compute the square root term:

$$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{(2a\cos\theta)^2 + (-2a\sin\theta)^2} = \sqrt{4a^2(\cos^2\theta + \sin^2\theta)} = \sqrt{4a^2} = 2a$$

So the formula simplifies to:

$$S = \int_0^{\frac{\pi}{2}} 2\pi(2a\cos\theta)\sin\theta \cdot 2a\, d\theta$$

$$S = 4\pi a \cdot 2a \int_0^{\frac{\pi}{2}} \cos\theta \sin\theta\, d\theta = 8\pi a^2 \int_0^{\frac{\pi}{2}} \cos\theta\sin\theta\, d\theta$$

Step 4: Evaluate the Integral

$$\int_0^{\frac{\pi}{2}} \cos\theta\sin\theta\, d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin(2\theta)\, d\theta = \frac{1}{2}\left[-\frac{1}{2}\cos(2\theta)\right]_0^{\frac{\pi}{2}} = \frac{1}{2}\left[-\frac{1}{2}(\cos\pi - \cos 0)\right]$$

$$= \frac{1}{2}\left[-\frac{1}{2}(-1-1)\right] = \frac{1}{2}\cdot\frac{1}{2}\cdot 2 = \frac{1}{2}$$

Step 5: Final Result

$$S = 8\pi a^2 \cdot \frac{1}{2} = \boxed{4\pi a^2}$$

✅ **Final Answer:**

The surface area of the solid formed by revolving the circle r=2acosθ about the x-axis is:

$$\boxed{4\pi a^2}$$

Q-3: (a)

Determine whether $\lim_{(x,y)\to(0,0)} \frac{x^6-y^2}{x^3y}$ exists, find it if exists. **(3 Marks)**

Answer:

Let's analyze the limit:

$$L = \lim_{(x,y)\to(0,0)} \frac{x^6-y^2}{x^3y}$$

We test the limit along **different paths**.

Path 1: Let y = x^3

$$\Rightarrow L = \lim_{x\to 0} \frac{x^6-x^6}{x^3\cdot x^3} = \frac{0}{x^6} = 0$$

Path 2: Let y = -x^3

$$L = \lim_{x\to 0} \frac{x^6-x^6}{x^3\cdot(-x^3)} = \frac{0}{-x^6} = 0$$

Path 3: Let y = x^k for general k

Then:

$$L = \lim_{x\to 0} \frac{x^6-x^{2k}}{x^3\cdot x^k} = \lim_{x\to 0} \frac{x^6-x^{2k}}{x^{k+3}} = \lim_{x\to 0} \left(x^{3-k} - x^{2k-(k+3)}\right)$$

This depends on the value of k. If k=1, then:

$$L = \lim_{x \to 0} \frac{x^6 - x^2}{x^4} = \lim_{x \to 0} (x^2 - \frac{1}{x^2}) \to -\infty$$

Thus, **limit does not exist** because it depends on the path.

✅ **Final Answer:**

Limit does not exist

Q-3: (b)

Applying the chain rule find $\frac{dw}{dt}$ $if\ w = x^2y - y^2$, where $x = sint$ and $y = e^t$ also find $\frac{dw}{dt}$ at $t = 0$

(4 Marks)

Answer:

We are given:

- $w = x^2 y - y^2$
- $x = \sin t, y = e^t$

We want:

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt}$$

Step 1: Partial Derivatives

Given $w = x^2 y - y^2$

$$\frac{\partial w}{\partial x} = 2xy, \quad \frac{\partial w}{\partial y} = x^2 - 2y$$

Step 2: Derivatives of x and y

$$\frac{dx}{dt} = \cos t, \quad \frac{dy}{dt} = e^t$$

Step 3: Apply Chain Rule

$$\frac{dw}{dt} = (2xy) \cdot \cos t + (x^2 - 2y) \cdot e^t$$

Now substitute $x = \sin t$, $y = e^t$:

$$\frac{dw}{dt} = 2\sin t \cdot e^t \cdot \cos t + (\sin^2 t - 2e^t) \cdot e^t$$

Step 4: Evaluate at t=0

At t=0:

- sint =0, cost=1, e^t = 1

Now compute:

$$\frac{dw}{dt} = 2(0)(1)(1) + (0^2 - 2(1))(1) = 0 + (-2)(1) = \boxed{-2}$$

✅ **Final Answer:**

$$\frac{dw}{dt} = 2\sin t \cdot e^t \cdot \cos t + (\sin^2 t - 2e^t) \cdot e^t$$

and at t=0,

$$\boxed{\frac{dw}{dt} = -2}$$

Q-3: (c) Find the extreme value of the function $f(x,y) = x^3 + y^3 + 3x^2 - 3y^2$ **(7 Marks)**

Answer:

Step 1: Find Critical Points

To find extreme values, compute the **first partial derivatives** and set them to **zero**:

$$f_x = \frac{\partial f}{\partial x} = 3x^2 + 6x, \quad f_y = \frac{\partial f}{\partial y} = 3y^2 - 6y$$

Set both equal to zero:

$$3x^2 + 6x = 0 \Rightarrow 3x(x+2) = 0 \Rightarrow x = 0, -2$$

$$3y^2 - 6y = 0 \Rightarrow 3y(y-2) = 0 \Rightarrow y = 0, 2$$

Step 2: Critical Points

The critical points are all combinations:

- (0,0), (0,2), (−2,0), (−2,2)

Step 3: Second Derivative Test

Compute second partial derivatives:

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = 6x + 6, \quad f_{yy} = \frac{\partial^2 f}{\partial y^2} = 6y - 6, \quad f_{xy} = f_{yx} = 0$$

Use the **discriminant**:

$$D = f_{xx} f_{yy} - (f_{xy})^2$$

Test each point:

1. **At** $(0, 0)$:

$$f_{xx} = 6,\ f_{yy} = -6,\ D = 6 \cdot (-6) - 0 = -36 < 0 \Rightarrow \text{Saddle point}$$

2. **At** $(0, 2)$:

$$f_{xx} = 6,\ f_{yy} = 6,\ D = 36 > 0,\ f_{xx} > 0 \Rightarrow \text{Local minimum}$$

$$f(0, 2) = 0 + 8 + 0 - 12 = -4$$

3. **At** $(-2, 0)$:

$$f_{xx} = -6,\ f_{yy} = -6,\ D = 36 > 0,\ f_{xx} < 0 \Rightarrow \text{Local maximum}$$

$$f(-2, 0) = (-8) + 0 + 12 - 0 = 4$$

4. **At** $(-2, 2)$:

$$f_{xx} = -6,\ f_{yy} = 6,\ D = -36 < 0 \Rightarrow \text{Saddle point}$$

✅ **Final Answer:**

- **Local minimum** at (0,2) with value −4

- **Local maximum** at (−2,0) with value 4
- Saddle points at (0,0) and (−2,2)

OR

Q-3: (a) If $u = \tan^{-1}(y/x)$ prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ **(3 Marks)**

Answer:

Given:

$$u = \tan^{-1}\left(\frac{y}{x}\right)$$

This is a function of two variables x and y.

Let's first find first-order partial derivatives, then second-order, and finally verify the identity.

Step 1: First-Order Partial Derivatives

$$u = \tan^{-1}\left(\frac{y}{x}\right) \Rightarrow \text{Let } z = \frac{y}{x} \Rightarrow u = \tan^{-1}(z)$$

Then, using chain rule:

- $\frac{\partial u}{\partial x} = \frac{1}{1+z^2} \cdot \frac{\partial}{\partial x}\left(\frac{y}{x}\right) = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \left(-\frac{y}{x^2}\right)$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{-y}{x^2+y^2}$$

Similarly,

$$\frac{\partial u}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \left(\frac{1}{x}\right) = \frac{x}{x^2+y^2}$$

Step 2: Second-Order Partial Derivatives

Now compute:

- $\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{-y}{x^2+y^2}\right)$

Here, treat y as constant:

$$\frac{\partial^2 u}{\partial x^2} = -y \cdot \frac{\partial}{\partial x}\left(\frac{1}{x^2+y^2}\right) = -y \cdot \left(\frac{-2x}{(x^2+y^2)^2}\right) = \frac{2xy}{(x^2+y^2)^2}$$

- $\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right)$

Treat x as constant:

$$\frac{\partial^2 u}{\partial y^2} = x \cdot \frac{\partial}{\partial y}\left(\frac{1}{x^2+y^2}\right) = x \cdot \left(\frac{-2y}{(x^2+y^2)^2}\right) = \frac{-2xy}{(x^2+y^2)^2}$$

Step 3: Add Both

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{2xy - 2xy}{(x^2+y^2)^2} = \boxed{0}$$

✅ **Final Answer:**

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \boxed{0}$$

Q-3: (b)

Find the tangent plane and normal line of the surface
$f(x, y, z) = x^2 + y^2 + z - 9 = 0$ at point $(1, 2, 4)$ **(4 Marks)**

Answer:

We are given the surface implicitly as:

$$F(x, y, z) = x^2 + y^2 + z - 9 = 0$$

To find:

1. **Tangent plane** at (1,2,4)
2. **Normal line** at (1,2,4)

Step 1: Gradient Vector (Normal Vector)

The gradient vector $\nabla F = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right)$ gives the **normal vector** to the surface.

$$\frac{\partial F}{\partial x} = 2x, \quad \frac{\partial F}{\partial y} = 2y, \quad \frac{\partial F}{\partial z} = 1$$

At the point (1,2,4):

$$\nabla F=(2\cdot 1,\ 2\cdot 2,\ 1)=(2,\ 4,\ 1)$$

Step 2: Tangent Plane Equation

General equation of tangent plane at point (x0,y0,z0):

$$\left.\frac{\partial F}{\partial x}\right|_{(P)} (x - x_0) + \left.\frac{\partial F}{\partial y}\right|_{(P)} (y - y_0) + \left.\frac{\partial F}{\partial z}\right|_{(P)} (z - z_0) = 0$$

Substitute:

$$2(x-1) +4 (y-2) + 1(z-4) =0$$

Simplify:

$$2x-2+4y-8+z-4=0 \Rightarrow 2x+4y+z=142$$

✅ **Tangent plane equation:**

$$2x+4y+z=14$$

Step 3: Normal Line Equation

Direction vector of normal = $\nabla F=(2,\ 4,\ 1)$
Passes through point (1,2,4)

So the parametric form of the normal line is:

$$x=1+2t,\ y=2+4t,\ z=4+t$$

✅ **Normal line equations:**

$$x=1+2t, y=2+4t, z=4+t$$

✅ **Final Answer:**

- **Tangent Plane:** $2x+4y+z=14$
- **Normal Line:** $x=1+2t,\ y=2+ 4t,\ z= 4+t$

Q-3: (c)

Applying Lagrange multipliers to determine the dimension of a rectangular box, open at the top, having volume 32 ft^3, and requiring the least amount of material for its construction.

(7 Marks)

Answer:

Step 1: Define Variables and Objective

Let:

- x = length,
- y = width,
- h = height of the open box.

Objective: Minimize Surface Area

Since the **top is open**, surface area is:

S = Base area +Lateral areas= xy+ 2xh+ 2yh

✅ So, minimize:

$$S(x, y, h) = xy+2xh+2yh$$

Constraint: Volume is 32

$$V = xyh = 32$$

So the constraint is:

$$g(x, y, h) = xyh - 32 = 0$$

Step 2: Use Lagrange Multipliers

Define:

$$\mathcal{L}(x, y, h, \lambda) = xy + 2xh + 2yh - \lambda(xyh - 32)$$

Compute partial derivatives:

1. $\frac{\partial \mathcal{L}}{\partial x} = y + 2h - \lambda(yh) = 0$
2. $\frac{\partial \mathcal{L}}{\partial y} = x + 2h - \lambda(xh) = 0$
3. $\frac{\partial \mathcal{L}}{\partial h} = 2x + 2y - \lambda(xy) = 0$
4. $\frac{\partial \mathcal{L}}{\partial \lambda} = xyh - 32 = 0$

Step 3: Solve the System

From equations (1) and (2):

(1): $y+2h=\lambda y h$
(2): $x+2h=\lambda x h$

Divide both sides of (1) and (2) by h (assume $h\neq0$:

$$\frac{y}{h} + 2 = \lambda y, \quad \frac{x}{h} + 2 = \lambda x$$

Now subtract these two:

$$\left(\frac{y}{h} + 2\right) - \left(\frac{x}{h} + 2\right) = \lambda y - \lambda x \Rightarrow \frac{y-x}{h} = \lambda(y-x)$$

If $x \neq y$, divide both sides:

$$\frac{1}{h} = \lambda \Rightarrow \text{contradiction unless } x = y$$

✅ So we must have: **$x=y$**

Now use Equation (3):

$$2x + 2y = \lambda xy \Rightarrow 4x = \lambda x^2 \Rightarrow \lambda = \frac{4}{x}$$

From earlier, we also had:

$$\lambda = \frac{1}{h} \Rightarrow \frac{4}{x} = \frac{1}{h} \Rightarrow h = \frac{x}{4}$$

Use Volume Constraint:

$$xyh = 32, \quad \text{but } x = y,\ h = \frac{x}{4} \Rightarrow x^2 \cdot \frac{x}{4} = 32 \Rightarrow \frac{x^3}{4} = 32 \Rightarrow x^3 = 128 \Rightarrow x = \sqrt[3]{128} = 2 \cdot \sqrt[3]{16}$$

$$x = y = \boxed{\sqrt[3]{128}} \approx 5.04, \quad h = \frac{x}{4} = \boxed{\frac{\sqrt[3]{128}}{4}} \approx 1.26$$

✅ **Final Answer:**

- $x = y = \boxed{\sqrt[3]{128}}$ ft
- $h = \boxed{\frac{\sqrt[3]{128}}{4}}$ ft

- Minimum surface area occurs when the base is square and height is one-fourth the side.

Q-4: (a)

State sandwich theorem for the sequence and discuss the convergence of the sequence $\left\{\frac{\cos n}{n}\right\}$

(3 Marks)

Answer:

Sandwich Theorem (Squeeze Theorem):

If for all n beyond some point,

$$a_n \le b_n \le c_n \quad \text{and} \quad \lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$$

then

$$\lim_{n\to\infty} b_n = L$$

Apply to Sequence $\left\{\frac{\cos n}{n}\right\}$

We know:

- $-1 \le \cos n \le 1$ for all n

$$\Rightarrow \frac{-1}{n} \le \frac{\cos n}{n} \le \frac{1}{n}$$

As $n \to \infty$, both bounds tend to 0:

$$\lim_{n\to\infty} \frac{-1}{n} = 0, \quad \lim_{n\to\infty} \frac{1}{n} = 0$$

So, by Sandwich Theorem:

$$\boxed{\lim_{n\to\infty} \frac{\cos n}{n} = 0}$$

✅ **Convergent**, limit is 0

Q-4: (b)

Test the convergence of the series $\sum_{n=0}^{\infty} \left(\frac{4^n+3}{5^n}\right)$, also find the sum of the series if it is convergent. **(4 Marks)**

Answer:

Let:

$$a_n = \frac{4n+3}{5^n}$$

We use the **Ratio Test**:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{4(n+1)+3}{5^{n+1}} \cdot \frac{5^n}{4n+3} = \frac{4n+7}{4n+3} \cdot \frac{1}{5}$$

As n→∞:

$$\frac{4n+7}{4n+3} \to 1 \Rightarrow \text{Limit} = \frac{1}{5} < 1$$

✅ **By Ratio Test**, the series **converges absolutely**.

To Find the Sum:

Let's try to split the series:

$$\sum_{n=0}^{\infty} \frac{4n+3}{5^n} = 4\sum_{n=0}^{\infty} \frac{n}{5^n} + 3\sum_{n=0}^{\infty} \frac{1}{5^n}$$

Use standard results:

- $\sum_{n=0}^{\infty} \frac{1}{r^n} = \frac{1}{1-\frac{1}{5}} = \frac{5}{4}$
- $\sum_{n=1}^{\infty} \frac{n}{5^n} = \frac{5}{(5-1)^2} = \frac{5}{16}$

- → So, shift index to start at n=1, adjust accordingly.

Now compute:

$$\sum_{n=0}^{\infty} \frac{n}{5^n} = 0 + \frac{1}{5} + \frac{2}{25} + \cdots = \frac{5}{16}$$

Then:

$$\sum = 4 \cdot \frac{5}{16} + 3 \cdot \frac{5}{4} = \frac{20}{16} + \frac{15}{4} = \frac{5}{4} + \frac{15}{4} = \frac{20}{4} = \boxed{5}$$

✅ **Final Answer:**

- **Series converges absolutely**
- **Sum** = 5

Q-4: (c)

Applying appropriate test find the interval of convergence and radius of convergence of the series $\sum_{n=0}^{\infty} \frac{(x-5)^n}{n^2}$ **(7 Marks)**

Answer:

This is a power series centered at x=5:

$$\sum_{n=0}^{\infty} \frac{(x-5)^n}{n^2}$$

Step 1: Use Ratio Test

Let:

$$a_n = \frac{(x-5)^n}{n^2}$$

Then,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-5)^{n+1}}{(n+1)^2}\cdot\frac{n^2}{(x-5)^n}\right| = |x-5|\cdot\lim_{n\to\infty}\left(\frac{n^2}{(n+1)^2}\right) =$$

$|x-5|\cdot 1=|x-5|$

By **Ratio Test**:
Converges when $|x-5|<1$,
Diverges when $|x-5|>1$

✅ So the **radius of convergence** is:

$$R=1$$

Step 2: Test Endpoints

(i) $x=5-1=4$:

$$\sum_{n=1}^{\infty}\frac{(4-5)^n}{n^2} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

This is an **alternating, convergent** series (converges absolutely).

(ii) $x=5+1=6$:

$$\sum_{n=1}^{\infty}\frac{(6-5)^n}{n^2} = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

Also **convergent** (p-series with $p=2>1$)

✅ So both endpoints are included.

✅ **Final Answer:**

- **Radius of convergence**: 1
- **Interval of convergence**: [4, 6]

OR

Q-4: (a) Investigate the convergence of the series $\sum_{n=0}^{\infty}\frac{4^n n! n!}{(2n!)}$ **(3 Marks)**

Answer:

Let:

$$a_n = \frac{4^n \cdot (n!)^2}{(2n)!}$$

We use the **Ratio Test**:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left(\frac{4^{n+1}(n+1)!(n+1)!}{(2n+2)!} \cdot \frac{(2n)!}{4^n n! n!} \right)$$

Simplify step-by-step:

- $\frac{4^{n+1}}{4^n} = 4$
- $\frac{(n+1)!^2}{n!^2} = (n+1)^2$
- $\frac{(2n)!}{(2n+2)!} = \frac{1}{(2n+2)(2n+1)}$

Putting it together:

$$= 4 \cdot (n+1)^2 \cdot \frac{1}{(2n+2)(2n+1)}$$

$$= \frac{4(n+1)^2}{(2n+2)(2n+1)} = \frac{4(n+1)^2}{4(n+1)(2n+1)} = \frac{(n+1)}{(2n+1)}$$

Now take the limit as n→∞:

$$\lim_{n\to\infty} \frac{n+1}{2n+1} = \frac{1}{2} < 1$$

✅ So, by **Ratio Test**, the series **converges absolutely**.

✅ **Final Answer:**

The series $\sum_{n=0}^{\infty} \frac{4^n n! n!}{(2n)!}$
converges absolutely by Ratio Test.

Convergent

Q-4: (b) Discuss the convergence of the series and find it's sum $\sum_{n=0}^{\infty} \frac{1}{n(n+1)}$ **(4 Marks)**

Answer:

We are given:

$$\sum_{n=0}^{\infty} \frac{1}{n(n+1)}$$

But note: the term at n=0 is undefined (division by zero).

✅ So, we **start the series from n=1**:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

Step 1: Use Partial Fractions

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

So the series becomes:

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right)$$

Step 2: Write First Few Terms

$$= \left(\frac{1}{1} - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \cdots$$

This is a **telescoping series**: most terms cancel out.

So we are left with:

$$S_n = 1 - \frac{1}{n+1}$$

Take the limit:

$$\lim_{n \to \infty} S_n = 1 - 0 = \boxed{1}$$

✅ **Final Answer:**

- The series is **convergent**
- The **sum is**: 1

Q-4: (c)

Applying appropriate test find the interval of convergence and radius of convergence of the series $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{3^n(n+1)}$ **(7 Marks)**

Answer:

Let:

$$a_n = \frac{(-1)^n x^n}{3^n(n+1)}$$

Step 1: Apply the Ratio Test

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+1} x^{n+1}}{3^{n+1}(n+2)} \cdot \frac{3^n(n+1)}{(-1)^n x^n}\right|$$

Simplify:

- $\frac{(-1)^{n+1}}{(-1)^n} = -1$ (absolute value = 1)
- $\frac{x^{n+1}}{x^n} = x$
- $\frac{3^n}{3^{n+1}} = \frac{1}{3}$

So:

$$= |x| \cdot \frac{1}{3} \cdot \frac{n+1}{n+2} \to \frac{|x|}{3} \cdot 1 = \frac{|x|}{3}$$

Conclusion from Ratio Test:

- Series **converges absolutely** if $|x|/3<1 \Rightarrow |x|<3$
- Series **diverges** if $|x|>3$

✅ So, **radius of convergence** is:

$$R=3$$

Step 2: Check Endpoints

At $x=3$:

$$\sum \frac{(-1)^n \cdot 3^n}{3^n(n+1)} = \sum \frac{(-1)^n}{n+1}$$

This is an **alternating series**, decreasing, and tends to 0 → **Converges**

At x=−3:

$$\sum \frac{(-1)^n(-3)^n}{3^n(n+1)} = \sum \frac{(-1)^n \cdot (-1)^n}{n+1} = \sum \frac{1}{n+1} = \sum_{n=1}^{\infty} \frac{1}{n}$$

This is the **harmonic series** → **Diverges**

✅ **Final Answer:**

- **Radius of Convergence**: 3
- **Interval of Convergence**: (−3, 3]

Q-5: (a)

Evaluate $\iint_R y^2 x \, dA$ over the rectangular region
$R = \{(x, y): -3 \leq x \leq 2, \ 0 \leq y \leq 1\}$ **(3 Marks)**

Answer:

Since the limits are constants, this is a **rectangular region**. We integrate in the order: **dy dx** or **dx dy**.

Let's integrate with respect to **y first**.

Step 1: Setup the Integral

$$\int_{x=-3}^{2} \int_{y=0}^{1} y^2 x \, dy \, dx$$

Step 2: Integrate w.r.t. y

$$= \int_{-3}^{2} x \left[\frac{y^3}{3}\right]_0^1 dx = \int_{-3}^{2} x \cdot \frac{1}{3} dx = \frac{1}{3}\int_{-3}^{2} x\, dx$$

$$= \frac{1}{3}\left[\frac{x^2}{2}\right]_{-3}^{2} = \frac{1}{3}\left(\frac{4}{2} - \frac{9}{2}\right) = \frac{1}{3} \cdot \left(-\frac{5}{2}\right) = \boxed{-\frac{5}{6}}$$

✅ **Final Answer:**

$$\boxed{-\frac{5}{6}}$$

Q-5: (b)

Use double integration to determine the area of the region R inclosed between the parabola $y = \frac{1}{2}x^2$, and the line $y = 2x$, also sketch the region. **(4 Marks)**

Answer:

Step 1: Find Points of Intersection

Set:

$$\frac{1}{2}x^2 = 2x \Rightarrow x^2 = 4x \Rightarrow x(x-4) = 0 \Rightarrow x = 0,\ 4$$

So the curves intersect at x=0 and x=4

Step 2: Area Between Curves

$$\text{Area} = \int_{x=0}^{4} [\text{upper} - \text{lower}]\, dx = \int_0^4 \left(2x - \frac{1}{2}x^2\right) dx$$

Step 3: Integrate

$$\int_0^4 \left(2x - \frac{1}{2}x^2\right) dx = \left[x^2 - \frac{1}{6}x^3\right]_0^4 = 16 - \frac{64}{6} = 16 - \frac{32}{3} = \frac{48-32}{3} = \boxed{\frac{16}{3}}$$

✅ **Final Answer:**

- **Area** enclosed = 16/3

Q-5: (c)

Applying change of the order of the integration to evaluate $\int_0^2 \int_{\frac{y}{2}}^1 e^{x^2}\, dx\, dy$, also sketch the region. **(7 Marks)**

Answer:

Step 1: Sketch and Describe the Region

The original limits are:

- Outer: y=0 to 2
- Inner: x=y to 2

So region R lies **below the line x=2**, **above the line x=y**, and between y=0 and y=2.

Sketch steps:

- Draw line x=y (diagonal)
- Draw vertical line x=2
- Shade region between x=y and x=2 for y∈[0,2]

Step 2: Change the Order of Integration

In original form: y→ x

Now, we switch to integrating **x first**:

From the sketch:

- x goes from **0 to 2**
- For each x, y goes from **0 to x**

✅ New limits:

$$\int_{x=0}^{2} \int_{y=0}^{x} e^{x^2}\, dy\, dx$$

Step 3: Evaluate the Integral

Now,

$$\int_{x=0}^{2}\int_{y=0}^{x} e^{x^2}\,dy\,dx$$

Since e^{x^2} is **independent of y**:

$$= \int_{x=0}^{2} e^{x^2}\cdot[y]_{y=0}^{x}\,dx = \int_0^2 x\cdot e^{x^2}\,dx$$

Now apply substitution:

Let $u = x^2 \Rightarrow du = 2xdx \Rightarrow xdx = \frac{1}{2}du$

When $x = 0$, $u = 0$; when $x = 2$, $u = 4$

$$= \int_{u=0}^{4} e^u\cdot\frac{1}{2}du = \frac{1}{2}\int_0^4 e^u du = \frac{1}{2}\left[e^u\right]_0^4 = \frac{1}{2}(e^4 - e^0) = \frac{1}{2}(e^4 - 1)$$

✅ **Final Answer:**

$$\boxed{\frac{1}{2}(e^4 - 1)}$$

OR

Q-5: (a) Evaluate $\int_0^1\int_x^1\int_0^{y-x} dzdydx$ **(3 Marks)**

Answer:

We evaluate the **triple integral** step-by-step.

Step 1: Integrate with respect to z

$$\int_{z=0}^{y-x} dz = (y - x) - 0 = y - x$$

Now the integral becomes:

$$\int_{x=0}^{1}\int_{y=x}^{1}(y - x)\,dy\,dx$$

Step 2: Integrate w.r.t. y

First, fix x, integrate:

$$\int_{y=x}^{1} (y-x)dy$$

Let's simplify:
Let u= y −x ⇒du = dy, when y= x⇒ u=0, when y= 1⇒ u=1−x

So:

$$\int_{y=x}^{1} (y-x)dy = \int_{0}^{1-x} u du = \left[\frac{u^2}{2}\right]_0^{1-x} = \frac{(1-x)^2}{2}$$

Step 3: Integrate w.r.t. x

Now:

$$\int_{x=0}^{1} \frac{(1-x)^2}{2}dx = \frac{1}{2}\int_0^1 (1-2x+x^2)dx = \frac{1}{2}\left[x - x^2 + \frac{x^3}{3}\right]_0^1 = \frac{1}{2}\left(1-1+\frac{1}{3}\right)$$

=1/2·1/3=1/6

✅ **Final Answer:**

1/6

Q-5: (b)

Evaluate the following integral by changing in to the polar co ordinate.
$\iint_R e^{x^2+y^2} dA$ where R is the semi circular region bounded by the x-axis and the curve $y = \sqrt{1-x^2}$ **(4 Marks)**

Answer:

Step 1: Understand the Region R

- y=1−x^2 is the **upper semicircle** of the unit circle x^2 + y^2 = 1
- So, R is the upper half-disk of radius 1 centered at origin.

Step 2: Convert to Polar Coordinates

Use substitutions:

- $x = r\cos\theta,\ y = r\sin\theta$
- $x^2 + y^2 = r^2$
- $dA = r\,dr\,d\theta$

New Limits in Polar Form:

- r∈[0,1]
- θ∈[0,π](upper half)

Step 3: Rewrite and Integrate

$$\iint_R e^{x^2+y^2}\,dA = \int_{\theta=0}^{\pi}\int_{r=0}^{1} e^{r^2}\cdot r\,dr\,d\theta$$

Step 4: Integrate w.r.t. r

Let u=r^2⇒du=2r dr⇒r dr=1/2du

When r=0, u=0; when r=1, u=1

$$\int_0^1 e^{r^2} r\,dr = \frac{1}{2}\int_0^1 e^u\,du = \frac{1}{2}(e^1 - e^0) = \frac{1}{2}(e-1)$$

Now multiply by angular integral:

$$\int_0^{\pi} d\theta = \pi \Rightarrow \text{Total Integral} = \pi\cdot\frac{1}{2}(e-1) = \boxed{\frac{\pi}{2}(e-1)}$$

✅ **Final Answer:**

$$\boxed{\frac{\pi}{2}(e-1)}$$

Q-5: (c)

A thin plate covers the triangular region bounded by the x-axis and the line $x = 1$ and $y = 2x$ in the first quadrant. The plate density at the point (x, y) is $\delta(x, y) = 6x + 6y + 6$ find the plate's mass, first moments and the centre of mass about the coordinate axis. **(7 Marks)**

Answer:

Step 1: Define the Region

- Triangle with:
  - Base: $x \in [0,1]$
  - Top edge: $y=2x$
- So limits are:

$$0 \le x \le 1, 0 \le y \le 2x$$

Step 2: Mass of the Plate

$$M = \iint_R \delta(x,y)\,dA = \int_{x=0}^{1} \int_{y=0}^{2x} (6x+6y+6)\,dy\,dx$$

Integrate w.r.t. y:

$$= \int_0^1 \left[6xy + 3y^2 + 6y\right]_0^{2x} dx = \int_0^1 \left[6x(2x) + 3(4x^2) + 6(2x)\right] dx =$$

$$\int_0^1 (12x^2 + 12x^2 + 12x)\,dx = \int_0^1 (24x^2 + 12x)dx$$

$$= \left[8x^3 + 6x^2\right]_0^1 = 8 + 6 = \boxed{14}$$

Step 3: First Moments

About y-axis:

$$M_y = \iint x \cdot \delta(x,y)\,dA = \int_0^1 \int_0^{2x} x(6x+6y+6)dy\,dx = \int_0^1 \left[6x^2y + 3xy^2 + 6xy\right]_0^{2x} dx$$

Substitute $y = 2x$:

$$= \int_0^1 \left(6x^2(2x) + 3x(4x^2) + 6x(2x)\right) dx = \int_0^1 (12x^3 + 12x^3 + 12x^2)dx = \int_0^1 (24x^3 + 12x^2)$$

$$= \left[6x^4 + 4x^3\right]_0^1 = 6 + 4 = \boxed{10}$$

About x-axis:

$$M_x = \iint y \cdot \delta(x,y)\, dA = \int_0^1 \int_0^{2x} y(6x+6y+6)dy\, dx = \int_0^1 \left[6x \cdot \frac{y^2}{2} + 2y^3 + 3y^2\right]_0^{2x} dx$$

Substitute $y = 2x$:

$$= \int_0^1 \left(6x \cdot \frac{4x^2}{2} + 2 \cdot 8x^3 + 3 \cdot 4x^2\right) dx = \int_0^1 (12x^3 + 16x^3 + 12x^2)dx = \int_0^1 (28x^3 + 12x^2)$$

$$= \left[7x^4 + 4x^3\right]_0^1 = 7 + 4 = \boxed{11}$$

Step 4: Center of Mass

$$\bar{x} = \frac{M_y}{M} = \frac{10}{14} = \boxed{\frac{5}{7}}, \quad \bar{y} = \frac{M_x}{M} = \frac{11}{14}$$

✅ **Final Answer:**

- **Mass** = 14
- **First moments**: My=10, Mx=11
- **Center of mass**:

$$\boxed{\left(\frac{5}{7}, \frac{11}{14}\right)}$$