GUJARAT TECHNOLOGICAL UNIVERSITY

BE-1 SEMESTER– PAPER SOLUTION – SUMMER 2025

Subject Name & Code:

Mathematics-I BE01000041

Q-1: (a)

Investigate the convergence of $\int_0^1 \frac{1}{1-x} dx$. **(3 Marks)**

Answer:

The given integral is:

$$\int_0^1 \frac{1}{1-x} \, dx$$

This is an **improper integral** because the integrand becomes **infinite** as x→1−.

Let's define the integral as:

$$\lim_{a \to 1^-} \int_0^a \frac{1}{1-x} \, dx$$

We use the substitution method or integrate directly:

$$\int \frac{1}{1-x} dx = -\ln|1-x| + C$$

Now apply limits:

$$\lim_{a \to 1^-} [-\ln|1-x|]_0^a = \lim_{a \to 1^-} [-\ln(1-a) + \ln(1-0)]$$

$$= \lim_{a \to 1^-} [-\ln(1-a) + \ln(1)] = \lim_{a \to 1^-} -\ln(1-a)$$

As $a \to 1^-$, $(1-a) \to 0^+$, so $\ln(1-a) \to -\infty$, thus:

$$-\ln(1-a) \to +\infty$$

✅ **Conclusion:**

Since the limit goes to **infinity**, the integral

$$\int_0^1 \frac{1}{1-x}\,dx$$

diverges.

Q-1: (b)

Define beta and gamma functions. What is the relationship between beta and gamma functions? **(4 Marks)**

Answer:

Definition of Gamma Function:

The **Gamma function**, denoted by Γ(n), is defined for n>0 as:

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\,dx$$

🔸 **Properties:**

- Γ(n+1)=nΓ(n)
- Γ(1)=1
- Γ(n)=(n−1) for natural number n

🔹 **Definition of Beta Function:**

The **Beta function**, denoted by B(m,n), is defined for m>0,n>0 as:

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

🔁 **Relationship between Beta and Gamma Functions:**

The relationship between the Beta and Gamma functions is given by:

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

✅ This is **a very important identity** in calculus and is often used to evaluate integrals involving rational powers.

Q-1: (c)

The region between the curve
$y = \sqrt{x}, 0 \leq x \leq 4$ and the x-axis is revolved about the x-axis to generate a solid. Find its volume. **(7 Marks)**

Answer:

We are asked to find the **volume of revolution** of the curve $y = \sqrt{x}$ from x=0 to x=4 about the **x-axis**.

We use the **disk method**:

$$V = \pi \int_a^b [f(x)]^2 \, dx$$

Here,

- $f(x) = \sqrt{x}$
- $a = 0, b = 4$

So,

$$V = \pi \int_0^4 (\sqrt{x})^2 \, dx = \pi \int_0^4 x \, dx$$

Now compute the integral:

$$V = \pi \left[\frac{x^2}{2} \right]_0^4 = \pi \left(\frac{4^2}{2} - \frac{0^2}{2} \right) = \pi \cdot \frac{16}{2} = 8\pi$$

✅ **Final Answer:**

$$\boxed{V = 8\pi \text{ units}^3}$$

Q-2: (a)

Evaluate $\lim_{x \to 0} \frac{3x - sinx}{x}$ using L'Hospital rule. **(3 Marks)**

Answer:

We are given:

$$\lim_{x \to 0} \frac{3x - \sin x}{x}$$

Check the form:

- As x→0: numerator → 3x−sinx→0
- Denominator → x→0

So it's an **indeterminate form of type** 0/0
We can apply **L'Hospital's Rule**, which says:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \quad \text{if } \frac{0}{0} \text{ or } \frac{\infty}{\infty}$$

Differentiate numerator and denominator:

- $f(x) = 3x - \sin x \Rightarrow f'(x) = 3 - \cos x$
- $g(x) = x \Rightarrow g'(x) = 1$

Apply L'Hospital's Rule:

$$\lim_{x \to 0} \frac{3 - \cos x}{1} = 3 - \cos(0) = 3 - 1 = 2$$

✅ **Final Answer:**

Q-2: (b)

Find the Taylor series generated by $f(x) = e^x$ at $x = 2$. **(4 Marks)**

Answer:

The **Taylor series** of a function f(x) centered at a is:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

Here:

- f(x)= e^x
- a=2

Now compute derivatives:

- f(x)= e^x
- f'(x)= e^x'
- f''(x)= e^x
- f(n)(x) = e^x, for all n

So at x=2x = 2x=2, all derivatives become f(n)(2) = e^2

Now substitute into the Taylor series:

$$e^x = e^2 + e^2(x-2) + \frac{e^2}{2!}(x-2)^2 + \frac{e^2}{3!}(x-2)^3 + \cdots$$

Factor out e^2:

$$e^x = e^2\left[1 + (x-2) + \frac{(x-2)^2}{2!} + \frac{(x-2)^3}{3!} + \cdots\right]$$

✅ **Final Answer:**

$$\boxed{e^x = e^2 \sum_{n=0}^{\infty} \frac{(x-2)^n}{n!}}$$

Q-2: (c)

Find the Maclaurin series for $f(x) = (1+x)^k$ where k is any real number. Using it, find the Maclaurin series for the function $\frac{1}{1-x}$.

(7 Marks)

Answer:

Solution Part 1: Maclaurin Series of (1 + x)^k

A **Maclaurin series** is a Taylor series centered at a=0.
The generalized binomial expansion for (1 + x)^k is:

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$$

Where:

$$\binom{k}{n} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}, \quad \text{for } n \geq 1, \quad \binom{k}{0} = 1$$

So,

$$(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$$

This is valid for |x|<1.

🧮 **Solution Part 2: Maclaurin Series for 1/1−x**

Let's use the result above.
We take $f(x) = (1+x)^k$, and set $k = -1$:

$$\frac{1}{1-x} = (1+(-x))^{-1}$$

Apply the binomial expansion:

$$(1-x)^{-1} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

So:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

This is valid for |x|<1

✅ **Final Answer:**

1. Maclaurin series for $(1+x)^k$:

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$$

2. Maclaurin series for $\frac{1}{1-x}$:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

OR

Q-2: (c)

Find the local extreme values of the function
$f(x) = 3x^4 - 2x^3 - 6x^2 + 6x + 1.$ **(7 Marks)**

Answer:

To find local extrema (local maxima or minima), we:

1. Find the **first derivative** f'(x)
2. Solve f'(x)=0 to get **critical points**
3. Use the **second derivative test** to classify those points

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🔹 **Step 1: First Derivative**

$$f(x) = 3x^4 - 2x^3 - 6x^2 + 6x + 1$$

Differentiate:

$$f'(x) = 12x^3 - 6x^2 - 12x + 6$$

🔹 **Step 2: Solve f'(x)=0**

$$12x^3 - 6x^2 - 12x + 6 = 0$$

Divide entire equation by 6:

$$2x^3 - x^2 - 2x + 1 = 0$$

Try Rational Root Theorem (try x=1):

$$2(1)^3 - (1)^2 - 2(1) + 1 = 2 - 1 - 2 + 1 = 0 \quad \Rightarrow x = 1 \text{ is a root}$$

Now divide polynomial by (x−1):

Using polynomial division or synthetic division:

$$2x^3 - x^2 - 2x + 1 = (x - 1)(2x^2 + x - 1)$$

Factor further:

$$2x^2 + x - 1 = (2x - 1)(x + 1)$$

So critical points are:

$$x = 1, \quad x = \frac{1}{2}, \quad x = -1$$

◆ **Step 3: Second Derivative Test**

$$f''(x) = \frac{d}{dx}(12x^3 - 6x^2 - 12x + 6) = 36x^2 - 12x - 12$$

Now evaluate f''(x) at each critical point:

- At x=−1:

$$f''(-1) = 36(-1)^2 - 12(-1) - 12 = 36 + 12 - 12 = 36 > 0 \Rightarrow \text{Local minimum}$$

- At $x = \frac{1}{2}$:

$$f''\left(\frac{1}{2}\right) = 36\left(\frac{1}{4}\right) - 12\left(\frac{1}{2}\right) - 12 = 9 - 6 - 12 = -9 < 0 \Rightarrow \text{Local maximum}$$

- At $x = 1$:

$$f''(1) = 36(1)^2 - 12(1) - 12 = 36 - 12 - 12 = 12 > 0 \Rightarrow \text{Local minimum}$$

✅ **Final Answer:**

- **Local minimum** at $x = -1$, $f(-1) = 3(-1)^4 - 2(-1)^3 - 6(-1)^2 + 6(-1) + 1 = 3 + 2 - 6 - 6 + 1 = -6$
- **Local maximum** at $x = \frac{1}{2}$, $f\left(\frac{1}{2}\right) = 3\left(\frac{1}{16}\right) - 2\left(\frac{1}{8}\right) - 6\left(\frac{1}{4}\right) + 6\left(\frac{1}{2}\right) + 1 = \frac{3}{16} - \frac{1}{4} - \frac{3}{2} + 3 + 1 = \frac{67}{16}$
- **Local minimum** at $x = 1$, $f(1) = 3 - 2 - 6 + 6 + 1 = 2$

Q-3: (a) Test the the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ for convergence or divergence. **(3 Marks)**

Answer:

Compare with the **p-series**:

$$\sum \frac{1}{n^p} \text{ converges if } p > 1$$

Note:

$$\frac{1}{n^2+1} < \frac{1}{n^2} \quad \text{and} \quad \sum \frac{1}{n^2} \text{ is a convergent p-series with } p = 2 > 1$$

So we apply the **comparison test**:

- $0 < \frac{1}{n^2+1} < \frac{1}{n^2}$
- Since $\sum \frac{1}{n^2}$ converges, so does $\sum \frac{1}{n^2+1}$

✅ **Final Answer:**

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^2+1} \text{ converges by comparison with } \sum \frac{1}{n^2}}$$

Q-3: (b) Test the convergence of the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$. **(4 Marks)**

Answer:

This is an **alternating series**, also known as the **alternating harmonic series**:

$$\sum_{n=1}^{\infty}(-1)^{n+1}\cdot\frac{1}{n}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$$

We apply the **Leibniz Test** (Alternating Series Test):

A series of the form $\sum(-1)^n a_n$ **converges** if:

1. $a_n > 0$
2. $a_{n+1} \leq a_n$ (i.e., decreasing)
3. $\lim_{n\to\infty} a_n = 0$

Let $a_n = \frac{1}{n}$

- ✅ $a_n > 0$
- ✅ a_n is decreasing because $\frac{1}{n+1} < \frac{1}{n}$
- ✅ $\lim_{n\to\infty} \frac{1}{n} = 0$

All conditions of the **Alternating Series Test** are satisfied.

Final Answer:

$$\boxed{\sum_{n=1}^{\infty}(-1)^{n+1}\cdot\frac{1}{n}\text{ converges by the Alternating Series Test}}$$

Q-3: (c)

Define absolutely convergent series and conditionally convergent series.
Investigate the convergence of the series $\sum_{n=1}^{\infty}\frac{\sin n}{n^2}$. **(7 Marks)**

Answer:

Definitions:

🔹 **Absolutely Convergent Series:**

A series ∑an is said to be **absolutely convergent** if the series of absolute values

∑|an| converges

🔹 **Conditionally Convergent Series:**

A series ∑an is said to be **conditionally convergent** if:

- ∑an converges, but
- ∑|an| **diverges**

🧮 **Investigation of** $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$

Let $a_n = \frac{\sin n}{n^2}$

We want to check convergence of the series:

Note:

- $|\sin n| \leq 1 \Rightarrow |\frac{\sin n}{n^2}| \leq \frac{1}{n^2}$

So:

$$\left|\frac{\sin n}{n^2}\right| \leq \frac{1}{n^2}$$

And we know:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \text{ is a convergent } p\text{-series } (p = 2 > 1)$$

By the **Comparison Test**:

$$\sum_{n=1}^{\infty} \left|\frac{\sin n}{n^2}\right| \text{ converges}$$

➡️ Therefore, $\sum \frac{\sin n}{n^2}$ is **absolutely convergent**.

✅ **Final Answer:**

- **Definition:**
  - A series ∑an is **absolutely convergent** if ∑|an| converges.
  - It is **conditionally convergent** if ∑an converges but ∑|an| diverges.

- **Conclusion:**

$$\boxed{\sum_{n=1}^{\infty} \frac{\sin n}{n^2} \text{ is absolutely convergent}}$$

OR

Q-3: (a) Investigate the convergence of the series $\sum_{n=1}^{\infty} \frac{2^n}{n!}$. **(3 Marks)**

Answer:

We are given the series:

$$\sum_{n=1}^{\infty} \frac{2^n}{n!}$$

This is a **series with positive terms**, so we can use the **Ratio Test**.

◆ **Ratio Test:**

Let $a_n = \frac{2^n}{n!}$

Compute:

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} = \lim_{n\to\infty} \frac{2 \cdot 2^n}{(n+1) \cdot n!} \cdot \frac{n!}{2^n} = \lim_{n\to\infty} \frac{2}{n+1}$$

Now evaluate the limit:

$$\lim_{n\to\infty} \frac{2}{n+1} = 0$$

Since L=0<1, the series **converges absolutely** by the Ratio Test.

✅ **Final Answer:**

$$\boxed{\sum_{n=1}^{\infty} \frac{2^n}{n!} \text{ converges absolutely by the Ratio Test}}$$

Q-3: (b) Find the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$. **(4 Marks)**

Answer:

We are given:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

This is a **telescoping series**. First, decompose the term using **partial fractions**:

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$$

Multiply both sides by n(n+1):

$$1=A(n+1)+B(n)$$

Now solve for A and B:

- Let n=−1: 1=A (0) +B (−1) ⇒B=−1
- Let n=1: 1=A (2) +B (1) ⇒1=2A−1⇒A=1

So:

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

◆ **Now write the series:**

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

Write first few terms:

$$\left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots$$

This is a **telescoping series** — most terms cancel out:

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) = 1$$

Why? Because:

$$\sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+1} \right) = 1 - \frac{1}{N+1} \xrightarrow{N \to \infty} 1$$

✅ **Final Answer:**

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1}$$

Q-3: (c) For what values of x does the series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ converge? **(7 Marks)**

Answer:

We are given the power series:

$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$$

This is a **Taylor series centered at** x=3. Let:

$$a_n = \frac{(x-3)^n}{n}$$

We want to find values of xxx for which the series converges. Use the **Ratio Test**:

🔹 **Step 1: Apply the Ratio Test**

Let:

$$a_n = \frac{(x-3)^n}{n}, \quad a_{n+1} = \frac{(x-3)^{n+1}}{n+1}$$

Then:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-3)^{n+1}}{n+1} \cdot \frac{n}{(x-3)^n}\right| = |x-3| \cdot \frac{n}{n+1}$$

Now take the limit:

$$\lim_{n\to\infty} |x-3| \cdot \frac{n}{n+1} = |x-3| \cdot \lim_{n\to\infty} \frac{n}{n+1} = |x-3| \cdot 1 = |x-3|$$

The Ratio Test says the series **converges if** the limit < 1.

- So, convergence condition is:

$$|x-3| < 1 \quad \Rightarrow \quad 2 < x < 4$$

🔹 **Step 2: Test the endpoints of the interval**

We now check x=2 and x=4

Case 1: x=2

$$\sum_{n=1}^{\infty} \frac{(2-3)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

This is the **alternating harmonic series**, which is **conditionally convergent** ✅

Case 2: x=4

$$\sum_{n=1}^{\infty} \frac{(4-3)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$$

This is the **harmonic series**, which **diverges** ❌

✅ **Final Answer:**

The series converges for:

$$\boxed{x \in [2,4) \text{ (i.e., converges absolutely in } (2,4), \text{ and conditionally at } x=2)}$$

Or written more precisely:

$$\boxed{\text{The series converges for } 2 \le x < 4}$$

Q-4: (a) If $f(x,y) = \frac{xy^2}{x^2+y^4}$, does $\lim_{(x,y)\to(0,0)} f(x,y)$ exist? **(3 Marks)**

Answer:

We are asked to check whether the limit

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$$

exists.

To do this, try approaching (0,0) along different paths:

◆ **Path 1: Along x=0**

$$f(0,y) = \frac{0 \cdot y^2}{0 + y^4} = 0$$

◆ **Path 2: Along y=0**

$$f(x,0) = \frac{x \cdot 0^2}{x^2 + 0} = 0$$

◆ **Path 3: Along $y = \sqrt{x}$ (i.e., $x = y^2$)**

Substitute $x = y^2$ into the function:

$$f(y^2, y) = \frac{y^2 \cdot y^2}{(y^2)^2 + y^4} = \frac{y^4}{y^4 + y^4} = \frac{y^4}{2y^4} = \frac{1}{2}$$

🔎 **Comparison:**

- Along x=0 and y=0, the limit is **0**

- Along x = y^2, the limit is **1/2**

❌ **Conclusion:**

Since the limit depends on the path taken, it does **not** approach the same value from all directions.
Hence, the limit does not exist.

✅ **Final Answer:**

The limit does not exist

Q-4: (b) If $z = x^2y + 3xy^4$, where $x = \sin 2t$ and $y = \cos t$, find $\frac{dz}{dt}$ when $t = 0$. **(4 Marks)**

Answer:

We are given a **composite function**:

- z=z(x(t),y(t))
- Use the **chain rule** to find $\frac{dz}{dt}$:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

🔹 **Step 1: Compute partial derivatives**

Given:

$$z = x^2y + 3xy^4$$

- $\frac{\partial z}{\partial x} = 2xy + 3y^4$
- $\frac{\partial z}{\partial y} = x^2 + 12xy^3$

🔹 **Step 2: Compute derivatives of x and y**

Given:

- $x = \sin 2t \Rightarrow \frac{dx}{dt} = 2\cos 2t$
- $y = \cos t \Rightarrow \frac{dy}{dt} = -\sin t$

🔹 **Step 3: Evaluate at t=0**

At t=0:

- $x = \sin 0 = 0$
- $y = \cos 0 = 1$
- $\frac{dx}{dt} = 2\cos 0 = 2$
- $\frac{dy}{dt} = -\sin 0 = 0$

Now substitute into partial derivatives:

- $\frac{\partial z}{\partial x} = 2(0)(1) + 3(1)^4 = 3$
- $\frac{\partial z}{\partial y} = (0)^2 + 12(0)(1)^3 = 0$

Now apply chain rule:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt} = 3 \cdot 2 + 0 \cdot 0 = 6$$

✅ **Final Answer:**

$$\boxed{\frac{dz}{dt} = 6 \quad \text{at } t = 0}$$

Q-4: (c)

Find the extreme values of the function $f(x, y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$. **(7 Marks)**

Answer:

This is a **constrained optimization problem**, where we optimize f(x,y)=x^2+2y^2

Subject to g(x,y)=x^2+y^2−1=0

We’ll use the **method of Lagrange multipliers**.

🔹 Step 1: Lagrange's Equations

Define:

- $f(x,y) = x^2 + 2y^2$
- $g(x,y) = x^2 + y^2 - 1$

Lagrange's method says:

$$\nabla f = \lambda \nabla g$$

Compute gradients:

- $\nabla f = (2x, 4y)$
- $\nabla g = (2x, 2y)$

Set up the system:

$$\begin{aligned} 2x &= \lambda \cdot 2x \quad (1) \\ 4y &= \lambda \cdot 2y \quad (2) \\ x^2 + y^2 &= 1 \quad (3) \end{aligned}$$

🔹 Step 2: Solve the System

From (1):

$$2x = \lambda \cdot 2x \Rightarrow \begin{cases} \text{If } x \neq 0 \Rightarrow \lambda = 1 \\ \text{If } x = 0, \text{ continue} \end{cases}$$

From (2):

$$4y = \lambda \cdot 2y \Rightarrow \begin{cases} \text{If } y \neq 0 \Rightarrow \lambda = 2 \\ \text{If } y = 0, \text{ continue} \end{cases}$$

So for both (1) and (2) to be satisfied simultaneously, consider the possible cases:

🔹 Case 1: $x \neq 0$, $y \neq 0$

Then $\lambda = 1$ from (1), and $\lambda = 2$ from (2)
❌ Contradiction → This case is **not possible**

◆ **Case 2: x=0x = 0x=0**

Then from constraint:

$x^2 + y^2 = 1 \Rightarrow y = \pm 1$

Evaluate $f(x, y)$:

- $f(0, 1) = 0 + 2(1)^2 = 2$
- $f(0, -1) = 0 + 2(1)^2 = 2$

◆ **Case 3: y=0**

Then from constraint:
x^2=1⇒x=±1

Evaluate f(x,y):

- f(1,0)=1+0=1
- f(−1,0)=1+0=1

✅ **Final Answer:**

- **Maximum value**: f (0, ±1) =2
- **Minimum value**: f (±1,0) =1

Maximum =2, Minimum =1

OR

Q-4: (a)

Find the equation of the tangent plane at the point $(-2,1,-3)$ to the ellipsoid $\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3.$

(3 Marks)

Answer:

This is an **implicit surface**, so we'll treat it as a level surface:

$$F(x, y, z) = \frac{x^2}{4} + y^2 + \frac{z^2}{9}$$

Let:

F(x,y,z)=3F

The equation of the **tangent plane** to a surface F(x,y,z)=c at point (x0,y0,z0)) is:

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

◆ **Step 1: Compute partial derivatives**

$$F_x = \frac{d}{dx}\left(\frac{x^2}{4}\right) = \frac{x}{2}, \quad F_y = \frac{d}{dy}(y^2) = 2y, \quad F_z = \frac{d}{dz}\left(\frac{z^2}{9}\right) = \frac{2z}{9}$$

◆ **Step 2: Evaluate at (−2,1,−3)**

$$F_x(-2) = \frac{-2}{2} = -1, \quad F_y(1) = 2, \quad F_z(-3) = \frac{-6}{9} = -\frac{2}{3}$$

◆ **Step 3: Write equation of the tangent plane**

$$-1(x + 2) + 2(y - 1) - \frac{2}{3}(z + 3) = 0$$

Multiply through to simplify:

$$-(x + 2) + 2(y - 1) - \frac{2}{3}(z + 3) = 0$$

$$-x - 2 + 2y - 2 - \frac{2}{3}z - 2 = 0$$

Combine terms:

−x+2y−2/3z−6=0

Multiply the whole equation by 3 to eliminate the fraction:

−3x+6y−2z−18=0

✅ **Final Answer:**

$$\boxed{-3x + 6y - 2z = 18} \quad \text{or} \quad \boxed{3x - 6y + 2z = -18}$$

Q-4: (b)

Find the directional derivative of the function $f(x, y) = x^2y^3 - 4y$ at the point $(2, -1)$ in the direction of the vector $\boldsymbol{v} = 2\boldsymbol{i} + 5\boldsymbol{j}$. **(4 Marks)**

Answer:

The **directional derivative** of f(x,y) at a point (x0,y0) in the direction of a vector $\vec{v}$ is:

$$D_{\vec{u}}f = \nabla f(x_0, y_0) \cdot \vec{u}$$

Where:

- $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$
- $\vec{u} = \frac{\vec{v}}{|\vec{v}|}$ is the **unit vector** in the direction of $\vec{v}$

🔹 **Step 1: Compute Gradient ∇ f**

Given:

$$f(x, y) = x^2y^3 - 4y$$

- $\frac{\partial f}{\partial x} = 2xy^3$
- $\frac{\partial f}{\partial y} = 3x^2y^2 - 4$

Now evaluate at (2,−1)):

- $\frac{\partial f}{\partial x} = 2(2)(-1)^3 = 4(-1) = -4$
- $\frac{\partial f}{\partial y} = 3(4)(1) - 4 = 12 - 4 = 8$

So:

$$\nabla f\,(2, -1) = (-4, 8)$$

🔹 **Step 2: Normalize the direction vector $\vec{v}$= (2,5)**

$$|\vec{v}| = \sqrt{2^2 + 5^2} = \sqrt{4 + 25} = \sqrt{29} \Rightarrow \vec{u} = \left(\frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}}\right)$$

🔹 **Step 3: Compute the dot product**

$$D_{\vec{u}} f = (-4, 8) \cdot \left(\frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}} \right) = \frac{1}{\sqrt{29}} (-4 \cdot 2 + 8 \cdot 5) = \frac{1}{\sqrt{29}} (-8 + 40) = \frac{32}{\sqrt{29}}$$

✅ **Final Answer:**

$$\boxed{\text{Directional derivative} = \frac{32}{\sqrt{29}}}$$

Q-4: (c)

Find the local extreme values of the function $f(x, y) = xy - x^2 - y^2 - 2x - 2y + 4$. **(7 Marks)**

Answer:

To find local extreme values:

1. Find **first partial derivatives** and set them to zero (critical points)
2. Use the **second derivative test** to classify the critical points

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🔹 **Step 1: First-order partial derivatives**

$$f(x, y) = xy - x^2 - y^2 - 2x - 2y + 4$$

- $f_x = \frac{\partial f}{\partial x} = y - 2x - 2$
- $f_y = \frac{\partial f}{\partial y} = x - 2y - 2$

Set both equal to zero:

$$\begin{cases} y - 2x - 2 = 0 & (1) \\ x - 2y - 2 = 0 & (2) \end{cases}$$

🔹 **Step 2: Solve the system**

From (1): y=2x+2

Substitute into (2):

$$x-2(2x+2)-2=0 \Rightarrow x-4x-4-2=0 \Rightarrow -3x=6 \Rightarrow x=-2$$

Then: $y=2(-2)+2=-4+2=-2$

So, critical point is $(-2,-2)$

◆ **Step 3: Second-order partial derivatives**

Compute second partials:

- $f_{xx} = \frac{\partial^2 f}{\partial x^2} = -2$
- $f_{yy} = \frac{\partial^2 f}{\partial y^2} = -2$
- $f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = 1$

Now compute the **discriminant**:

$$D = f_{xx} f_{yy} - (f_{xy})^2 = (-2)(-2) - (1)^2 = 4 - 1 = 3 > 0$$

- $D>0$
- $f_{xx}=-2<0$

So, this point is a **local maximum**

◆ **Step 4: Evaluate f(−2,−2)**

$$f(-2,-2)=(-2)(-2)-(-2)2-(-2)2-2(-2)-2(-2)+4=4-4-4+4+4+4=8$$

✅ **Final Answer:**

- **Local maximum at** $(-2,-2)$
- **Maximum value**: 8

Q-5: (a) Evaluate $\int_0^3 \int_0^2 (4-y^2)\,dy\,dx$ **(3 Marks)**

Answer:

This is a **double integral** over a rectangle:

- 0≤x≤2
- 0≤y≤3

Note that the inner integral is with respect to y, so integrate 4 - y^2 first:

◆ **Step 1: Inner integral**

$$\int_0^3 (4 - y^2)\, dy = \left[4y - \frac{y^3}{3}\right]_0^3 = \left(4 \cdot 3 - \frac{27}{3}\right) - 0 = 12 - 9 = 3$$

◆ **Step 2: Outer integral**

Now integrate with respect to xxx:

$$\int_0^2 3\, dx = 3x\Big|_0^2 = 3 \cdot 2 = 6$$

✅ **Final Answer:**

6

Q-5: (b)

Evaluate $\iint_R (3x + 4y^2) dA$ where R is the region in the upper half-plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$. **(4 Marks)**

Answer:

This is best solved in **polar coordinates**, since the region is **bounded by circles** and symmetric about the origin.

◆ **Step 1: Understand the region**

- The region RRR lies **between two concentric circles** with radii 1 and 2
- **Only the upper half-plane**, so θ∈[0,π]
- Radius r∈[1,2]

◆ **Step 2: Change to polar coordinates**

Recall:

- x=r cosθ
- y=r sinθ
- dA=r dr dθ

The integrand becomes:

$$3x + 4y^2 = 3r\cos\theta + 4(r\sin\theta)^2 = 3r\cos\theta + 4r^2\sin^2\theta$$

Now write the double integral:

$$\iint_R (3x + 4y^2)\, dA = \int_0^\pi \int_1^2 \left(3r\cos\theta + 4r^2\sin^2\theta\right) r\, dr\, d\theta$$

Multiply r inside:

$$= \int_0^\pi \int_1^2 \left(3r^2\cos\theta + 4r^3\sin^2\theta\right)\, dr\, d\theta$$

◆ **Step 3: Evaluate inner integral**

Break it into two parts:

I_1:

$$\int_1^2 3r^2\cos\theta\, dr = 3\cos\theta \int_1^2 r^2\, dr = 3\cos\theta \left[\frac{r^3}{3}\right]_1^2 = \cos\theta(8-1) = 7\cos\theta$$

I_2:

$$\int_1^2 4r^3\sin^2\theta\, dr = 4\sin^2\theta \int_1^2 r^3\, dr = 4\sin^2\theta \left[\frac{r^4}{4}\right]_1^2 = \sin^2\theta(16-1) = 15\sin^2\theta$$

So the total inner integral:

$$7\cos\theta + 15\sin^2\theta$$

◆ **Step 4: Integrate with respect to θ**

$$\int_0^\pi \left(7\cos\theta + 15\sin^2\theta\right)\, d\theta$$

Break into two integrals:

First term:

$$\int_0^{\pi} 7\cos\theta\, d\theta = 7[\sin\theta]_0^{\pi} = 7(0-0) = 0$$

Second term:

Use identity: $\sin^2\theta = \frac{1-\cos 2\theta}{2}$

$$\int_0^{\pi} 15\sin^2\theta\, d\theta = 15\int_0^{\pi} \frac{1-\cos 2\theta}{2}\, d\theta = \frac{15}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi} = \frac{15}{2}(\pi - 0) = \frac{15\pi}{2}$$

✅ **Final Answer:**

15π/2

Q-5: (c)

Sketch the region of integration, reverse the order of integration and evaluate the integral $\int_0^1 \int_x^1 \sin(y^2)\, dy dx$. **(7 Marks)**

Answer:

We are given:

- Outer integral: x∈[0,1]
- Inner integral: y∈[x,1]

So the region R is:

$$R = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 1,\ x \le y \le 1\}$$

This is the **triangle** bounded by:

- x=0
- y=1
- y=x

🧮 **Step 2: Reverse the Order of Integration**

Original:

- x∈ [0,1]
- y∈[x,1]

When reversing, we look at:

- y∈ [0,1
- x∈ [0, y]

So the new integral becomes:

$$\int_0^1 \int_0^y \sin(y^2)\, dx\, dy$$

🧮 **Step 3: Evaluate the Integral**

$$\int_0^1 \left[\int_0^y \sin(y^2)\, dx\right] dy$$

Since sin(y^2) is constant with respect to x, integrate:

$$\int_0^y \sin(y^2)\, dx = \sin(y^2) \cdot x\Big|_0^y = y \cdot \sin(y^2)$$

Now integrate with respect to y:

$$\int_0^1 y \cdot \sin(y^2)\, dy$$

Let's use substitution:

Let $u = y^2 \Rightarrow du = 2y\, dy \Rightarrow y\, dy = \frac{1}{2}du$

Change limits:

- When y=0⇒u=0
- When y=1⇒u=1

So:

$$\int_0^1 y\sin(y^2)\, dy = \frac{1}{2}\int_0^1 \sin(u)\, du = \frac{1}{2}[-\cos(u)]_0^1 = \frac{1}{2}(-\cos 1 + \cos 0)$$

$$= \frac{1}{2}(1 - \cos 1)$$

✅ **Final Answer:**

$$\boxed{\frac{1-\cos 1}{2}}$$

OR

Q-5: (a)

Evaluate the triple integral $\iiint_B xyz^2\, dV$ where B is the rectangular box given by
$B = \{(x, y, z): 0 \le x \le 1, -1 \le y \le 2, 0 \le z \le 3\}$. **(3 Marks)**

Answer:

We are integrating over a rectangular box. All limits are constants, so we integrate term by term:

$$\iiint_B xyz^2\, dV = \int_0^1 \int_{-1}^2 \int_0^3 xyz^2\, dz\, dy\, dx$$

◆ **Step 1: Integrate w.r.t. z**

$$\int_0^3 xyz^2\, dz = xy \int_0^3 z^2\, dz = xy \left[\frac{z^3}{3}\right]_0^3 = xy \cdot \frac{27}{3} = 9xy$$

◆ **Step 2: Integrate w.r.t. y**

$$\int_{-1}^2 9xy\, dy = 9x \int_{-1}^2 y\, dy = 9x \left[\frac{y^2}{2}\right]_{-1}^2 = 9x \cdot \left(\frac{4-1}{2}\right) = 9x \cdot \frac{3}{2} = \frac{27}{2}x$$

◆ **Step 3: Integrate w.r.t. x**

$$\int_0^1 \frac{27}{2} x\, dx = \frac{27}{2} \cdot \left[\frac{x^2}{2}\right]_0^1 = \frac{27}{2} \cdot \frac{1}{2} = \frac{27}{4}$$

✅ **Final Answer:**

27/4

Q-5: (b) Find the area of the region R bounded by $y = 2x^2$ and $y^2 = 4x$. **(4 Marks)**

Answer:

Step 1: Understand the region

We are given two curves:

1. $y = 2x^2$ → a **parabola** opening upward
2. $y^2 = 4x \Rightarrow x = \frac{y^2}{4}$ → a **sideways parabola**

Let's find the points of intersection:

Substitute $y = 2x^2$ into $y^2 = 4x$:

$$(2x^2)^2 = 4x \Rightarrow 4x^4 = 4x \Rightarrow x^4 = x \Rightarrow x(x^3 - 1) = 0$$

So:

- x=0, x=1

Find corresponding y values:

- At x=0: y=0
- At x=1x : y = 2(1)^2 = 2

🔹 **Step 2: Express area between curves**

Since y = 2x^2 is the upper curve and from y^2 = 4x, we get:

$$x = \frac{y^2}{4}$$

We integrate **with respect to** y from $y = 0$ to $y = 2$

Left curve: $x = \frac{y^2}{4}$

Right curve: solve $y = 2x^2 \Rightarrow x = \sqrt{y/2}$

Now area is:

$$A = \int_0^2 \left[\sqrt{\frac{y}{2}} - \frac{y^2}{4}\right] dy$$

Let's evaluate this:

Split it into two parts:

I_1: $\int_0^2 \sqrt{\frac{y}{2}}\, dy = \int_0^2 \frac{1}{\sqrt{2}} y^{1/2}\, dy = \frac{1}{\sqrt{2}} \cdot \frac{2}{3} y^{3/2} \Big|_0^2 = \frac{2}{3\sqrt{2}}(2)^{3/2} = \frac{2}{3\sqrt{2}} \cdot 2\sqrt{2} = \frac{4}{3}$

I_2: $\int_0^2 \frac{y^2}{4}\, dy = \frac{1}{4} \cdot \frac{y^3}{3} \Big|_0^2 = \frac{1}{4} \cdot \frac{8}{3} = \frac{2}{3}$

So:

$$A = \frac{4}{3} - \frac{2}{3} = \frac{2}{3}$$

✅ **Final Answer:**

2/3

Q-5: (c)

Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region R in the xy-plane bounded by the lines $y = 2x$ and the parabola $y = x^2$.

(7 Marks)

Answer:

Step 1: Sketch and Set Up Limits

We are to integrate z = x^2 + y^2 over the region bounded between:

- y = x^2 (a parabola opening upward)
- y=2x (a straight line)

Find points of intersection:

$$x^2 = 2x \Rightarrow x^2 - 2x = 0 \Rightarrow x(x-2) = 0 \Rightarrow x = 0, 2$$

So, region is between:

- x∈[0,2]
- For each x, y∈[x2,2x]

◆ **Step 2: Set up the integral**

$$V = \int_0^2 \int_{x^2}^{2x} (x^2 + y^2)\, dy\, dx$$

◆ **Step 3: Inner integral w.r.t. yyy**

$$\int_{x^2}^{2x} (x^2 + y^2)\, dy = x^2(y) + \frac{y^3}{3}\Big|_{x^2}^{2x} = x^2(2x - x^2) + \frac{(2x)^3 - (x^2)^3}{3} = x^2(2x - x^2) + \frac{8x^3 -}{3}$$

$$= 2x^3 - x^4 + \frac{8x^3 - x^6}{3}$$

Now combine:

$$= (2x^3 - x^4) + \left(\frac{8x^3 - x^6}{3}\right)$$

Let’s combine the terms into a single integral:

$$V = \int_0^2 \left[2x^3 - x^4 + \frac{8x^3 - x^6}{3}\right] dx$$

Write with common denominator:

$$= \int_0^2 \left(\frac{6x^3 - 3x^4 + 8x^3 - x^6}{3}\right) dx = \int_0^2 \left(\frac{14x^3 - 3x^4 - x^6}{3}\right) dx$$

◆ **Step 4: Integrate term by term**

$$\int_0^2 \frac{14x^3 - 3x^4 - x^6}{3} dx = \frac{1}{3}\left[\frac{14x^4}{4} - \frac{3x^5}{5} - \frac{x^7}{7}\right]_0^2$$

Evaluate:

- $\frac{14x^4}{4} = \frac{14 \cdot 16}{4} = 56$
- $\frac{3x^5}{5} = \frac{3 \cdot 32}{5} = \frac{96}{5}$
- $\frac{x^7}{7} = \frac{128}{7}$

So:

$$V = \frac{1}{3}\left(56 - \frac{96}{5} - \frac{128}{7}\right)$$

Compute inside:

Get common denominator (LCM of 5 and 7 is 35):

$$56 = \frac{1960}{35}, \quad \frac{96}{5} = \frac{672}{35}, \quad \frac{128}{7} = \frac{640}{35}$$

Now:

$$\frac{1}{3} \cdot \left(\frac{1960 - 672 - 640}{35}\right) = \frac{1}{3} \cdot \frac{648}{35} = \frac{648}{105}$$

Simplify:

$$\boxed{V = \frac{216}{35}}$$