Cyclicity & Factorial: Number System

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Cyclicity

1. Digits 0, 1, 5 & 6

Observation. These digits always retain the same unit (rightmost) digit when raised to any positive integer power.

$0^n = 0$

$1^n = 1$

$5^n = 5$

$6^n = 6$

Examples (unit digits):

$5^2 = 25$

$1^6 = 1$

$0^4 = 0$

$6^3 = 216$

This means no matter how large the power is, the unit digit never changes for these numbers. For example, 5 raised to any power always ends in 5. So if you see 5^{2000}, you immediately know the unit digit is 5 - no calculation needed.

2. Digits 4 & 9

Observation. Each of 4 and 9 has a cyclicity of length two for their unit digit.

For 4:

$4^2 = 16$

$4^3 = 64$

$4^4 = 256$

Unit digits alternate as 6, 4, 6, 4, ... so the pattern repeats every two powers (odd → 4, even → 6).

For 9:

$9^2 = 81$

$9^3 = 729$

Unit digits alternate as 1, 9, 1, 9, ... so the cycle length is two (odd → 9, even → 1).

4^{odd} = 4: If 4 is raised to the power of an odd number, then the unit digit will be 4.

4^{even} = 6: If 4 is raised to the power of an even number, then the unit digit will be 6.

9^{odd} = 9: If 9 is raised to the power of an odd number, then the unit digit will be 9.

9^{even} = 1: If 9 is raised to the power of an even number, then the unit digit will be 1.

Example: Find the unit digit of the following numbers:

1. $189^{562589743}$

Unit digit of base 189 is **9**.
Cycle for 9 is (9, 1), length 2.
The exponent 562589743 is **odd**.
Odd power → unit digit = 9.
Answer: 9

2. $279^{698745832}$

Unit digit of base 279 is **9**.
Cycle for 9 is (9, 1), length 2.
The exponent 698745832 is **even**.
Even power → unit digit = 1.
Answer: 1

3. $154^{258741369}$

Unit digit of base 154 is **4**.
Cycle for 4 is (4, 6), length 2.
The exponent 258741369 is **odd**.
Odd power → unit digit = 4.
Answer: 4

4. $194^{65478932}$

Unit digit of base 194 is **4**.
Cycle for 4 is (4, 6), length 2.
The exponent 65478932 is **even**.
Even power → unit digit = 6.
Answer: 6

3. Digits 2, 3, 7 & 8

Observation. Each of these digits has a cyclicity of length four for the unit digit.

For 2:

$2^1 = 2$

$2^2 = 4$

$2^3 = 8$

$2^4 = 16$

So the cycle of unit digits is 2, 4, 8, 6 and then repeats.

For 3:

$3^1 = 3$

$3^2 = 9$

$3^3 = 27$

$3^4 = 81$

So the cycle is 3, 9, 7, 1 and then repeats. Similar four-term cycles hold for 7 and 8.

Thus digits 2, 3, 7 and 8 repeat their unit digits every four powers.

Cyclicity Table

The patterns above are summarised in a cyclicity table (last-digit cycles depend only on the base's unit digit). A table is a quick reference to obtain the last digit of a large power without full evaluation.

Number	Cyclicity	Power Cycle
1	1	1
2	4	2, 4, 8, 6
3	4	3, 9, 7, 1
4	2	4, 6
5	1	5
6	1	6
7	4	7, 9, 3, 1
8	4	8, 4, 2, 6
9	2	9, 1
10	1	0

How to use this table -

Step 1: Look at the unit digit of your base number only (ignore all other digits).

Step 2: Find that digit in the left column. Read its cyclicity number (1, 2, or 4).

Step 3: Divide the exponent by the cyclicity number. The remainder tells you which position in the power cycle to pick.

Example: Find unit digit of 287^{562581}.

Unit digit of base = 7.

Cyclicity of 7 = 4. Cycle is (7, 9, 3, 1).

562581 ÷ 4 = remainder 1.

1st position in cycle = 7.

Answer: 7.

Remarks: digits that have cycle length 1 or 2 effectively repeat within every four exponents as well, so working modulo 4 (or modulo cycle length) is the standard approach to these problems.

> **Important:** If the remainder is 0 after dividing the exponent by the cycle length, do NOT use position 0. Use the last position in the cycle instead. Example: 2^8 → 8 ÷ 4 = remainder 0 → use 4th position in (2,4,8,6) = 6. Answer: 6.

MULTIPLE CHOICE QUESTION

Try yourself: Find the Unit digit of 287^{562581}

A 3

B 6

C 7

D 9

View Solution

Factors

Definition. A **factor** (or divisor) of a positive integer N is a positive integer that divides N exactly without leaving a remainder.

Example: Factors of 24 are 1, 2, 3, 4, 6, 8, 12, 24.

Number of Factors

If the prime factorisation of a positive integer N is

$N = p^a \times q^b \times r^c$

where p, q, r are distinct primes and a, b, c are positive integers, then the total number of positive divisors (factors) of N is

$(a+1)(b+1)(c+1)$

Example. Find the number of factors of $2^4 \times 3^2$.

$(4+1)(2+1) = 5 \times 3 = 15$

Tip: Every perfect square has an odd number of factors; a non-square has an even number of factors.

Number of Ways of Expressing a Given Number as a Product of Two Factors

Given the total number of positive divisors $d(N)$ of N:

If $d(N)$ is even, the number of unordered pairs (x, y) with $x \le y$ and $xy = N$ equals
$\frac{d(N)}{2}$

If $d(N)$ is odd (this happens only when N is a perfect square), the number of unordered distinct pairs (x, y) with \(x
$\frac{d(N)-1}{2}$
and the number of unordered pairs allowing equality $(x \le y)$ equals
$\frac{d(N)+1}{2}$

Examples.

(i) For 148: prime factorisation gives $148 = 2^2 \times 37$. Then $d(148) = (2+1)(1+1) = 6$, which is even. Therefore unordered pairs (x, y) with $xy = 148$ are $6/2 = 3$ ways.

product of two factors in 6/2 or 3 ways using .

(ii) For 144: $144 = 2^4 \times 3^2$. Then $d(144) = (4+1)(2+1) = 15$, which is odd (144 is a perfect square). Number of ways to write 144 as a product of two different numbers is $(15-1)/2 = 7$; number of ways allowing equal factors is $(15+1)/2 = 8$.

Sum of the Factors of a Number

If $N = a^p\, b^q\, c^r$ where a, b, c are distinct primes and p, q, r are positive integers, then the sum of all positive divisors of N is

$$\sigma(N) = \prod_{\text{primes } t \mid N} \frac{t^{(\text{power of } t) + 1} - 1}{t - 1}$$

Number of Co-primes to N (Euler's Totient)

Two numbers are co-prime if the only number that divides both of them is 1. 8 and 9 are co-prime (no shared factor). 6 and 9 are NOT co-prime (since both of them are divisible by 3).

$\varphi(N)$ counts how many numbers from 1 to N-1 are co-prime with N. The formula below calculates this without checking every number individually.

The number of positive integers less than N that are co-prime to N is given by Euler's totient function $\varphi(N)$.

If $N = \prod_{i=1}^{k} p_i^{a_i}$ where p_i are distinct primes, then

$\varphi(N) = N \prod_{i=1}^{k} (1 - \frac{1}{p_i})$

Factorial

Definition. For a positive integer n, the **factorial** $n!$ is the product of all positive integers from 1 to n.

$n! = 1 \times 2 \times 3 \times \cdots \times n$

Recurrence relation:

$n! = n \times (n-1)!$

Why do we divide repeatedly?

Think of 95! = 1×2×3×...×95. We want to count how many times 3 appears as a factor across all those numbers.

Multiples of 3 (like 3, 6, 9...) each give one factor of 3. There are ⌊95/3⌋ = 31 such numbers.

Multiples of 9 (like 9, 18, 27...) give an EXTRA factor of 3 (already counted once above).

There are ⌊95/9⌋ = 10 such numbers.

Multiples of 27 give yet another extra factor. ⌊95/27⌋ = 3.

Multiples of 81 give one more extra. ⌊95/81⌋ = 1.

Multiples of 243 would be next but 243 > 95, so we stop.

Total factors of 3 = 31 + 10 + 3 + 1 = 45. So 3^{45} is the highest power.

Example 1. Find the largest power of 3 that divides $95!$ exactly.

Sol.

Count how many factors of 3 appear in $95!$ by Legendre's formula: sum the integer parts of successive divisions of 95 by powers of 3.

$$\lfloor \frac{95}{3} \rfloor = 31$$
$$\lfloor \frac{95}{3^2} \rfloor = \lfloor \frac{95}{9} \rfloor = 10$$
$$\lfloor \frac{95}{3^3} \rfloor = \lfloor \frac{95}{27} \rfloor = 3$$
$$\lfloor \frac{95}{3^4} \rfloor = \lfloor \frac{95}{81} \rfloor = 1$$

Add the quotients to get the exponent of 3 in $95!$:

$$31 + 10 + 3 + 1 = 45$$

Therefore the highest power of 3 dividing $95!$ is

$$3^{45}$$

Remark: This method applies directly when the divisor is prime. For composite divisors, factor them into primes and use the prime exponents obtained by Legendre's formula to determine the composite power.

Example 2. Find the largest power of 12 that divides $200!$.

Sol.

Factorise 12:

$$12 = 2^2 \times 3$$

Find the exponent of 2 in 200! using Legendre's formula.

$\lfloor \frac{200}{2} \rfloor = 100$

$\lfloor \frac{200}{4} \rfloor = 50$ $\lfloor \frac{200}{8} \rfloor = 25$ $\lfloor \frac{200}{16} \rfloor = 12$ $\lfloor \frac{200}{32} \rfloor = 6$ $\lfloor \frac{200}{64} \rfloor = 3$ $\lfloor \frac{200}{128} \rfloor = 1$

Sum them to obtain the exponent of 2:

$100 + 50 + 25 + 12 + 6 + 3 + 1 = 197$

The exponent of 4 (which is 2^2) in $200!$ is

$\lfloor \frac{197}{2} \rfloor = 98$

Now find the exponent of 3 in $200!$:

$\lfloor \frac{200}{3} \rfloor = 66$ $\lfloor \frac{200}{9} \rfloor = 22$ $\lfloor \frac{200}{27} \rfloor = 7$ $\lfloor \frac{200}{81} \rfloor = 2$

Sum:

$66 + 22 + 7 + 2 = 97$

The limiting exponent for 12 in $200!$ is the smaller of the exponent of 4 (98) and the exponent of 3 (97), because one copy of **12 needs one 2^2 AND one 3.** We have enough 2^2 for 98 copies, but only enough 3 for 97 copies. The 3 runs out first, so 97 is the maximum. **The limiting ingredient sets the limit.**

Thus the largest power of 12 dividing $200!$ is

12^{97}

> **Note:** Why divide 197 by 2? Because 12 = 2^2 × 3. Each copy of 12 needs TWO factors of 2 (i.e., one 2^2). We found 197 total factors of 2 in 200!. Dividing by 2 tells us how many complete pairs of 2 we can form → 98 pairs. So at most 98 copies of 12 can come from the factor of 2^2 side.

Example 3. What is the last digit of $2^{34} \times 3^{34} \times 4^{34}$?

Sol.

Compute the last digit of each factor separately, then multiply the last digits and take the unit digit of the product.

Find the unit digit of 2^{34} . The cycle for base 2 is (2,4,8,6) of length 4; $34 \ mod \ 4 = 2$.

$\text{unit digit of } 2^{34} = 4$

Find the unit digit of 3^{34} . The cycle for base 3 is (3,9,7,1) of length 4; $34 \ mod \ 4 = 2$.

$\text{unit digit of } 3^{34} = 9$

Find the unit digit of 4^{34}. The cycle for base 4 is (4,6) of length 2; 34 is even.

$\text{unit digit of } 4^{34} = 6$

Multiply the unit digits: $4 \times 9 \times 6 = 216$.

The unit digit is 6.

Example 4. What is the rightmost non-zero digit of $(270)^{270}$?

Sol.

Remove powers of 10. Since $270 = 27 \times 10$, we have

$(270)^{270} = 27^{270} \times 10^{270}$

The trailing zeros are from 10^{270}; the rightmost non-zero digit equals the last digit of 27^{270}, i.e. the last digit of 7^{270}.

The cycle for 7 is (7,9,3,1) of length 4. Compute $270 \; mod \; 4 = 2$.

$\text{unit digit of } 7^{270} = 9$

Hence the rightmost non-zero digit of $(270)^{270}$ is 9.

Example 5. How many factors does 1296 have?

Sol.

Prime factorise 1296:

$1296 = 2^4 \times 3^4$

Number of factors:

$(4 + 1)(4 + 1) = 25$

Example 6. If x is the sum of all the factors of 3128, y is the number of factors of x, and z is the number of ways of writing y as a product of two numbers, find z.

Sol.

Factorise 3128:

$3128 = 2^3 \times 17 \times 23$

Sum of factors x using the product formula:

$x = (\frac{2^{3+1} - 1}{2 - 1})(\frac{17^{1+1} - 1}{17 - 1})(\frac{23^{1+1} - 1}{23 - 1})$

Compute the factors:

$\dfrac{2^4 - 1}{1} = 15, 17 + 1 = 18, 23 + 1 = 24$

$x = 15 \times 18 \times 24 = 6480$

Prime factorise x:

$6480 = 2^4 \times 3^4 \times 5$

Number of factors of x (that is y):

$y = (4 + 1)(4 + 1)(1 + 1) = 5 \times 5 \times 2 = 50$

Number of unordered ways to write $y = 50$ as a product of two positive integers equals half the number of divisors of 50. First find divisors count of 50:

$50 = 2 \times 5^2$

$d(50) = (1 + 1)(2 + 1) = 6$

Unordered pairs (a, b) with $ab = 50$:

\dfrac{6}{2}=3\)

Thus $z = 3$.

Example 7. How many co-primes are there to 240 that are less than 240?

Sol.

Factorise 240:

$240 = 2^4 \times 3 \times 5$

Use Euler's totient function:

$\varphi(240) = 240(1 - \frac{1}{2})(1 - \frac{1}{3})(1 - \frac{1}{5})$

=240\times\dfrac{1}{2}\times\dfrac{2}{3}\times\dfrac{4}{5}=64\)

So there are 64 integers less than 240 that are co-prime to 240.

Example 8. What is the sum of all co-primes to 748 that are less than 748?

Sol.

If a number k is co-prime to N, then (N-k) is also co-prime to N. They pair up symmetrically. Each pair sums to N, and there are $\dfrac{N\varphi(N)}{2}$ such pairs. So total sum = N × φ(N)/2.

Factorise 748:

$748 = 2^2 \times 11 \times 17$

Compute $\varphi(748)$:

$\varphi(748) = 748(1 - \frac{1}{2})(1 - \frac{1}{11})(1 - \frac{1}{17})$

=374\times\dfrac{10}{11}\times\dfrac{16}{17}=320\)

Sum of all integers less than N and co-prime to N equals $\dfrac{N\varphi(N)}{2}$ (when $N > 1$).

$\text{Sum} = \dfrac{748 \times 320}{2} = 374 \times 320 = 119680$

Example 9. In how many ways can 5544 be written as a product of two co-primes?

Sol.

Let N have n distinct prime factors. The number of ways to write N as a product of two co-prime positive integers equals 2^{n-1}.

Prime factorise 5544:

$5544 = 2^3 \times 3^2 \times 7 \times 11$

There are $n = 4$ distinct primes (2,3,7,11). Therefore the number of ways is

$2^{4-1} = 2^3 = 8$

Multiplying the Last Two Digits of the Exponents in Ten Seconds (Shortcuts and Tricks)

Learning cyclicity tables and the behaviour of last digits under powers allows quick mental calculation of last digits for large exponents; this is a common and useful shortcut for competitive exams.

Remember the cyclicity for digits 0-9 (use the cyclicity table). Once the cycle length for a base's unit digit is known, reduce the exponent modulo the cycle length to find the corresponding last digit.

Key trick: To find the last digit of a^n, determine the unit digit of a, find the cycle length for that unit digit, compute $n \, mod \, (\text{cycle length})$ (treat remainder 0 as the cycle length), and pick the corresponding term in the cycle.

Last digit / unit's digit of a^n