Chapter 5: Free and Damped Free Vibration / Torsional Vibration

Repeated Questions:

1. Damped Free Vibration Parameters (Numerical):

- **Q.** A damped system has a mass M. It is observed that it makes X oscillations in Y seconds. The amplitude reduces to Z of its initial value after W cycles. Determine: (i) Stiffness, (ii) Logarithmic Decrement, (iii) Damping Factor, (iv) Damping Coefficient. **(SUMMER 2022, 7 Marks; SUMMER 2024, 7 Marks; WINTER 2024, 7 Marks)**

Answer:

Given:

Mass m, X oscillations in Y seconds, amplitude reduces to Z of initial value after W cycles.

To Find:

(i) Stiffness k, (ii) Logarithmic decrement δ, (iii) Damping factor ζ, (iv) Damping coefficient c.

Formulae:

$\omega_n = \sqrt{(k/m)},\ f_n = 1/T = X/Y$

$\delta = (1/W)\ \ln(1/Z)$

$\zeta = \delta / \sqrt{(4\pi^2 + \delta^2)}$

$c = 2\zeta\sqrt{(km)} = 2\zeta m\omega_n$

Solution:

Natural frequency from observed oscillations:

$f_n = X/Y,\ \omega_n = 2\pi f_n$

Stiffness:

$k = m\omega_n^2$

Logarithmic decrement:

$\delta = (1/W)\ \ln(A_0/A_W) = (1/W)\ \ln(1/Z)$

Damping factor:

$\zeta = \delta / \sqrt{(4\pi^2 + \delta^2)}$

Damping coefficient:

$c = 2\zeta\sqrt{(km)}$

Final Answer:

$k = m\omega_n^2,\ \delta = (1/W)\ \ln(1/Z),\ \zeta = \delta / \sqrt{(4\pi^2 + \delta^2)},\ c = 2\zeta\sqrt{(km)}$

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2. Define Key Terms:

- **Q.** Define: Natural Frequency, Damping, Forced Vibration, Free Vibration, Resonance. **(SUMMER 2022, 3 Marks; WINTER 2022, 3 Marks; WINTER 2023, 3 Marks; SUMMER 2025, 3 Marks; WINTER 2025, 3 Marks)**

Answer:

Term	Definition
Natural Frequency	Frequency at which a system oscillates freely without external force or damping: f_n = (1/2π)√(k/m)
Damping	Resistance to motion that dissipates vibration energy, reducing amplitude over time.
Forced Vibration	Vibration produced when a system is subjected to an external periodic force.
Free Vibration	Vibration of a system under initial disturbance only, without external periodic force.
Resonance	Condition when excitation frequency equals natural frequency, causing maximum amplitude.

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3. Logarithmic Decrement & Damping Factor from Successive Amplitudes (Numerical):

- **Q.** The successive amplitudes of vibrations are A1, A2, A3… Determine the logarithmic decrement and damping factor. **(WINTER 2022, 7 Marks; WINTER 2023, 7 Marks)**

Answer:

Given: Successive amplitudes A_1, A_2, A_3, …

- **To Find:** Logarithmic decrement δ and damping factor ζ.

Formula:

$$\delta = \ln(A_1/A_2) = \ln(A_2/A_3) = \ldots$$

For n cycles:

$$\delta = (1/n)\ln(A_1/A_{(n+1)})$$

$$\zeta = \delta / \sqrt{(4\pi^2 + \delta^2)}$$

Solution:

Take any two successive amplitudes:

$$\delta = \ln(A_1/A_2)$$

If amplitudes after n cycles are given:

$$\delta = (1/n)\ln(A_1/A_{(n+1)})$$

Calculate damping factor:

$$\zeta = \delta / \sqrt{(4\pi^2 + \delta^2)}$$

For small damping:

$$\zeta \approx \delta/2\pi$$

Final Answer:

$$\delta = \ln(A_1/A_2),\ \zeta = \delta / \sqrt{(4\pi^2 + \delta^2)}$$

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4. Natural Frequency of Spring-Mass System (Numerical):

- **Q.** Find the natural frequency of the system shown in the figure. (Involves finding equivalent stiffness for combinations of springs.) [Diagram required – refer to the original question paper] **(WINTER 2022, 7 Marks; SUMMER 2023, 7 Marks; SUMMER 2025, 7 Marks OR)**

Answer:

General Method:

1. Find equivalent stiffness k_eq for series/parallel spring combinations.

Parallel: $k_{eq} = k_1 + k_2$

Series: $1/k_{eq} = 1/k_1 + 1/k_2$

2. Natural frequency:

$$f_n = (1/2\pi)\sqrt{(k_{eq}/m)}$$

Example (SUMMER 2026):

Given:

- Spring stiffness k = 350 N/m
- Mass m = 0.35 kg
- Initial displacement x_o = 20 mm
- Damping coefficient c = 14 N·s/m
- **To Find:** (a) Critical damping coefficient c_c, (b) Damped natural frequency f_d, (c) Logarithmic decrement δ.

Formula:

$$\omega_n = \sqrt{(k/m)},\ c_c = 2\sqrt{(km)} = 2m\omega_n$$
$$\zeta = c/c_c,\ \omega_d = \omega_n\sqrt{(1-\zeta^2)}$$
$$f_d = \omega_d/2\pi,\ \delta = 2\pi\zeta/\sqrt{(1-\zeta^2)}$$

Solution:

Natural frequency:

$$\omega_n = \sqrt{(350/0.35)} = \sqrt{1000} = 31.62\ \text{rad/s}$$

Critical damping coefficient:

$$c_c = 2\sqrt{(350(0.35))} = 2(11.07) = 22.14\ \text{N·s/m}$$

Damping factor:

$$\zeta = 14/22.14 = 0.632$$

Damped natural frequency:

$$\omega_d = 31.62\sqrt{(1-0.632^2)} = 31.62(0.775) = 24.51\ \text{rad/s}$$
$$f_d = 24.51/2\pi = 3.90\ \text{Hz}$$

Logarithmic decrement:

$$\delta = 2\pi(0.632)/\sqrt{(1-0.632^2)} = 5.13$$

Final Answer:

$$c_c = 22.14\ \text{N·s/m},\ f_d = 3.90\ \text{Hz},\ \delta = 5.13$$

Alternate Example (SUMMER 2026 OR):

m = 30 kg, k = 20 kN/m = 20000 N/m, ζ = 0.25

$$\omega_n = \sqrt{(20000/30)} = 25.82\ \text{rad/s}$$
$$c_c = 2\sqrt{(km)} = 2\sqrt{(20000(30))} = 1549.2\ \text{N·s/m}$$
$$\omega_d = 25.82\sqrt{(1-0.25^2)} = 25.0\ \text{rad/s}$$
$$\delta = 2\pi(0.25)/\sqrt{(1-0.25^2)} = 1.623$$

Ratio of successive amplitudes:

$$A_1/A_2 = e^{\delta} = e^{1.623} = 5.07$$

5. Types of Damping:

- **Q.** Compare the displacement-time curves for Underdamped, Overdamped, and Critically damped systems. **(WINTER 2022, 3 Marks; WINTER 2024, 4 Marks)**

Answer:

Type	Condition	Displacement–Time Curve
Underdamped	ζ < 1	Oscillatory decay with decreasing amplitude
Critically damped	ζ = 1	Fastest return to equilibrium without oscillation
Overdamped	ζ > 1	Slow non-oscillatory return to equilibrium

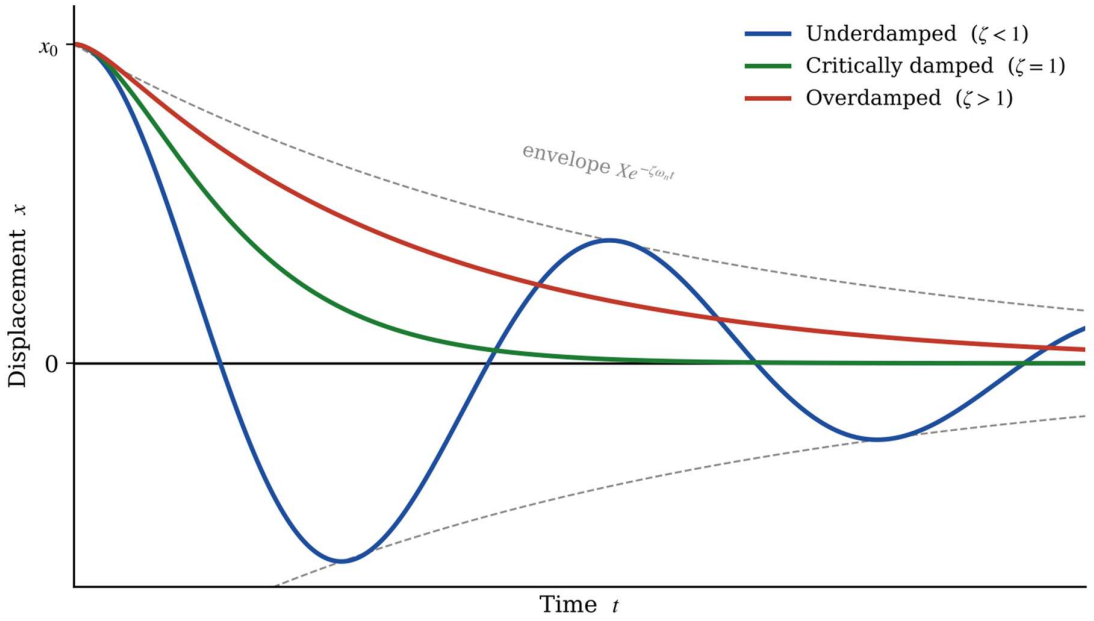


Fig. 15: Response curves for underdamped, critically damped and overdamped systems

- **Q.** List the types of damping and explain any one. / Explain the method of dry friction damping. **(WINTER 2023, 3 Marks; SUMMER 2025, 3 Marks; WINTER 2025, 4 Marks OR)**

Answer:

Types of damping:

- **Viscous damping:** Resistance proportional to velocity, F = cv.
- **Dry friction (Coulomb) damping:** Constant friction force opposing motion.
- **Structural/hysteretic damping:** Energy loss due to internal material friction.
- **Fluid damping:** Resistance due to fluid viscosity.

Dry friction damping:

- Friction force F = μN remains constant.
- Amplitude decreases linearly with time.
- Equation of motion: mẍ ± μN + kx = 0.
- Used in leaf springs, brake mechanisms.

6. Critically Damped System:

- **Q.** Derive equation for critically damped system. **(WINTER 2022, 4 Marks OR; SUMMER 2025, 4 Marks)**

Answer:

Equation of motion:

$$m\ddot{x} + c\dot{x} + kx = 0$$

$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = 0$$

For critically damped, $\zeta = 1$:

$$\ddot{x} + 2\omega_n \dot{x} + \omega_n^2 x = 0$$

Characteristic equation:

$$D^2 + 2\omega_n D + \omega_n^2 = 0$$

$$(D + \omega_n)^2 = 0$$

Roots:

$$D_1 = D_2 = -\omega_n$$

Solution:

$$x = (A + Bt)\, e^{(-\omega_n t)}$$

Initial conditions $x(0) = x_0$, $\dot{x}(0) = 0$:

$$A = x_0,\ B = \omega_n x_0$$

$$x = x_0\,(1 + \omega_n t)\, e^{(-\omega_n t)}$$

This is the critically damped response—no oscillation, fastest return to equilibrium.

Other Important Questions:

1. Torsional Vibration (Theory):

- **Q.** Derive an expression for the natural frequency of torsional vibration of a disc connected at the end of a shaft. **(WINTER 2023, 3 Marks)**

Answer:

Formula:

$$\omega_n = \sqrt{(k_t/I)},\ f_n = (1/2\pi)\sqrt{(k_t/I)}$$

where torsional stiffness:

$$k_t = GJ/L$$

For solid circular shaft:

$$J = \pi d^4/32$$

$$f_n = (1/2\pi)\sqrt{(G\pi d^4/(32LI))}$$

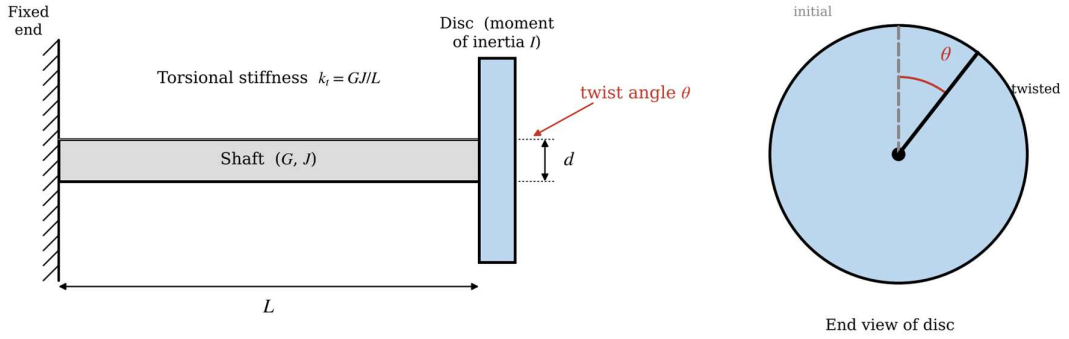


Fig. 16: Torsional vibration of a shaft carrying a disc

2. Torsional Vibration (Numerical):

- **Q.** Two rotors A and B are attached to a shaft. Find the position of the node and the natural frequency of torsional vibration. **(SUMMER 2022, 7 Marks)**

Answer:

Two-rotor system:

Given: I_1, I_2, shaft stiffness k_t.

Node position:

$$l_1/l_2 = I_2/I_1$$

Natural frequency:

$$\omega_n = \sqrt{(k_t\,(1/I_1 + 1/I_2))}$$

- **Q.** Analyze a three-rotor torsional vibration system and determine its natural frequencies. **(SUMMER 2026, 7 Marks)**

Answer:

Three-rotor system:

Set up equations:

$$I_1\ddot{\theta}_1 + k_t1\,(\theta_1 - \theta_2) = 0$$
$$I_2\ddot{\theta}_2 + k_t1\,(\theta_2 - \theta_1) + k_t2\,(\theta_2 - \theta_3) = 0$$
$$I_3\ddot{\theta}_3 + k_t2\,(\theta_3 - \theta_2) = 0$$

Solve determinant for natural frequencies.

- **Q.** A motor drives a centrifugal pump through gearing. The speed of the pump is one-third of the speed of the motor. The diameter of pinion shaft is 40 mm and it is 300 mm long. The mass moment of inertia of motor is 400 kg·m². The gear shaft, on which impeller of centrifugal pump is mounted, has 60 mm diameter and is 600 mm long. The mass moment of inertia of the impeller is 1500 kg·m². Neglecting the inertia of gear and pinion, determine the natural frequency of torsional vibration. **(SUMMER 2026, 7 Marks OR)**

Answer:

Given:

- Motor speed N_m, pump speed N_p = N_m/3
- Pinion shaft: d_p = 40 mm, L_p = 300 mm
- Motor inertia I_m = 400 kg·m²
- Gear shaft: d_g = 60 mm, L_g = 600 mm
- Impeller inertia I_i = 1500 kg·m²
- Neglect gear and pinion inertia.
- G for steel = 80 GPa.
- **To Find:** Natural frequency of torsional vibration.

Formula:

Equivalent inertia referred to motor shaft:

$$I_eq = I_m + I_i\,(N_p/N_m)^2 = I_m + I_i/9$$

Equivalent stiffness referred to motor shaft:

$$k_eq = [1/k_p + 1/(k_g\,(N_p/N_m)^2)]^{-1}$$

where

$$k_p = GJ_p/L_p,\ \ k_g = GJ_g/L_g$$

$J_p = \pi d_p^4/32,\ J_g = \pi d_g^4/32$

$\omega_n = \sqrt{(k_{eq}/I_{eq})}$

Solution:

$J_p = \pi(0.04)^4/32 = 2.513 \times 10^{-7}\ m^4$

$J_g = \pi(0.06)^4/32 = 1.272 \times 10^{-6}\ m^4$

$k_p = 80\times10^9\ (2.513\times10^{-7})/0.3 = 67013\ N\cdot m/rad$

$k_g = 80\times10^9\ (1.272\times10^{-6})/0.6 = 169600\ N\cdot m/rad$

Refer gear shaft to motor speed:

$k_g' = k_g\ (N_p/N_m)^2 = 169600\ (1/3)^2 = 18844\ N\cdot m/rad$

Equivalent stiffness:

$1/k_{eq} = 1/67013 + 1/18844$

$k_{eq} = 14700\ N\cdot m/rad$

Equivalent inertia:

$I_{eq} = 400 + 1500/9 = 400 + 166.67 = 566.67\ kg\cdot m^2$

Natural frequency:

$\omega_n = \sqrt{(14700/566.67)} = 5.09\ rad/s$

$f_n = 5.09/2\pi = 0.81\ Hz$

Final Answer:

$f_n = 0.81\ Hz$

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3. Logarithmic Decrement:

- **Q.** Define Logarithmic Decrement and derive expression of it with usual notations. **(WINTER 2025, 4 Marks)**

Answer:

- **Definition:** Natural logarithm of ratio of two successive amplitudes.

$\delta = \ln(A_1/A_2)$

Derivation:

For damped free vibration:

$x = X\ e^{(-\zeta\omega_n t)}\ \sin(\omega_d t + \varphi)$

At $t = t_1$, amplitude:

$A_1 = X\ e^{(-\zeta\omega_n t_1)}$

At $t = t_2 = t_1 + T_d$, where $T_d = 2\pi/\omega_d$:

$A_2 = X\ e^{(-\zeta\omega_n (t_1 + T_d))}$

Ratio:

$A_1/A_2 = e^{(\zeta\omega_n T_d)}$

Therefore:

$\delta = \zeta\omega_n T_d = 2\pi\zeta/\sqrt{(1 - \zeta^2)}$

For small damping:

$\delta \approx 2\pi\zeta$

- **Q.** Explain logarithmic decrement and its use. **(SUMMER 2026, 3 Marks)**

Answer:

- **Logarithmic decrement:** Natural logarithm of ratio of two successive amplitudes, $\delta = \ln(A_1/A_2)$.

Use:

- Determines damping factor from measured amplitudes.
- Used to find damping coefficient $c = 2\zeta\sqrt{(km)}$.

- Helps in vibration isolation and absorber design.

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4. Causes, Effects and Types of Vibration:

- **Q.** What are the causes and effects of vibrations? **(WINTER 2025, 3 Marks)**

Answer:

Causes:

- Unbalanced forces in rotating/reciprocating parts.
- External periodic forces.
- Elastic nature of materials.
- Earthquakes, wind, waves.

Effects:

- Noise and discomfort.
- Fatigue failure of components.
- Loosening of fasteners.
- Reduced machine life and accuracy.

- **Q.** Discuss briefly with neat sketches the longitudinal, transverse and torsion, free vibrations. **(WINTER 2025, 4 Marks)**

Answer:

Types of free vibration:

- **Longitudinal:** Particles move along length; e.g., spring-mass in axial direction.
- **Transverse:** Particles move perpendicular to length; e.g., beam bending vibration.
- **Torsional:** Particles rotate about axis; e.g., shaft-disc system.

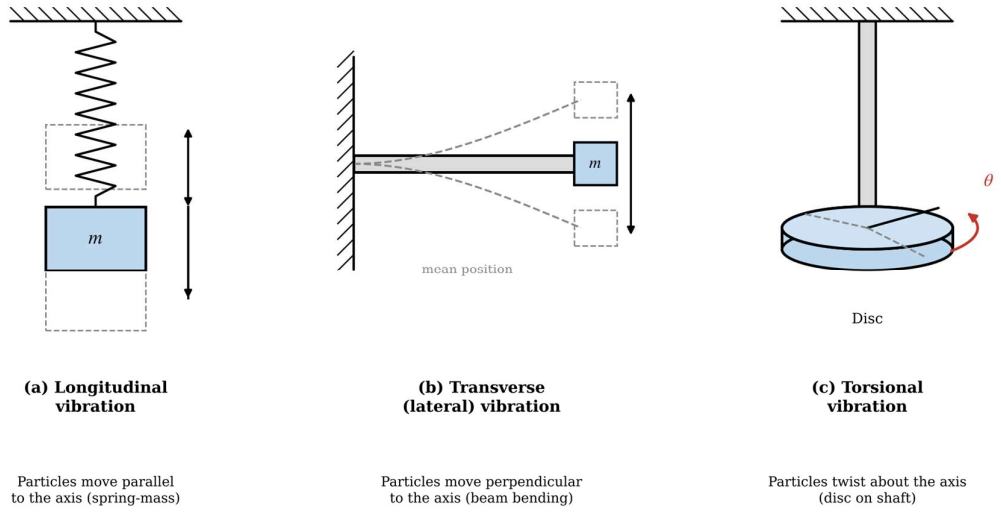


Fig. 17: Longitudinal, transverse and torsional vibrations

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5. Longitudinal and Transverse Vibration of a Shaft (Numerical):

- **Q.** A cantilever shaft 50 mm diameter and 300 mm long has a disc of mass 100 kg at its free end. The Young's modulus for the shaft material is 200 GN/m². Determine the frequency of longitudinal and transverse vibrations of the shaft. **(WINTER 2025, 7 Marks)**

Answer:

Given:

- Diameter d = 50 mm = 0.05 m
- Length L = 300 mm = 0.3 m
- Disc mass m = 100 kg
- Young's modulus E = 200 GN/m² = 200 × 10^9 N/m²
- **To Find:** Longitudinal frequency f_n,long and transverse frequency f_n,trans.

Formula:

$A = \pi d^2/4,\ k_{long} = EA/L$

$I = \pi d^4/64,\ k_{trans} = 3EI/L^3$

$f_n = (1/2\pi)\sqrt{(k/m)}$

Solution:

Area:

$A = \pi(0.05)^2/4 = 1.9635 \times 10^{-3}\ m^2$

Longitudinal stiffness:

$k_{long} = 200\times10^9\ (1.9635\times10^{-3})/0.3 = 1.309 \times 10^9\ N/m$

Longitudinal frequency:

$f_{n,long} = (1/2\pi)\sqrt{(1.309\times10^9/100)} = 575.8\ Hz$

Moment of inertia:

$I = \pi(0.05)^4/64 = 3.068 \times 10^{-7}\ m^4$

Transverse stiffness:

$k_{trans} = 3(200\times10^9)(3.068\times10^{-7})/(0.3)^3 = 6.82 \times 10^6\ N/m$

Transverse frequency:

$f_{n,trans} = (1/2\pi)\sqrt{(6.82\times10^6/100)} = 41.6\ Hz$

Final Answer:

$f_{n,long} = 575.8\ Hz,\ f_{n,trans} = 41.6\ Hz$

Chapter 6: Forced Vibration, Damping Response and Vibration Isolation

Repeated Questions:

1. Forced Vibration due to Rotating Unbalance (Numerical):

- **Q.** A machine of mass M is supported on springs of total stiffness K. It has a rotating unbalance force F at a speed of N rpm. If the damping factor is ζ, determine: (i) the amplitude of forced vibration, (ii) the transmissibility, and (iii) the force transmitted. **(WINTER 2022, 7 Marks; SUMMER 2023, 7 Marks; SUMMER 2025, 7 Marks)**

Answer:

Given:

Machine mass M, total stiffness K, unbalance mass m at eccentricity e, speed N rpm, damping factor ζ.

$\omega = 2\pi N/60,\ \omega_n = \sqrt{(K/M)},\ r = \omega/\omega_n$

$c_c = 2\sqrt{(KM)},\ c = \zeta c_c$

To Find:

(i) Amplitude of forced vibration X, (ii) Transmissibility ε, (iii) Force transmitted F_T.

Formulae:

$X = me\omega^2 / \sqrt{[(K - M\omega^2)^2 + (c\omega)^2]}$

$\varepsilon = F_T/F_0 = \sqrt{[(1 + (2\zeta r)^2) / ((1 - r^2)^2 + (2\zeta r)^2)]}$

$F_0 = me\omega^2,\ F_T = \varepsilon F_0$

Solution:

1. Calculate ω, ω_n, and frequency ratio r.
2. Calculate unbalance force amplitude $F_o = me\omega^2$.
3. Substitute in amplitude formula to find X.
4. Calculate damping coefficient $c = 2\zeta\sqrt{(KM)}$.
5. Calculate transmissibility ε using the formula.
6. Calculate transmitted force $F_T = \varepsilon F_o$.

Final Answer:

$X = me\omega^2 / \sqrt{[(K - M\omega^2)^2 + (c\omega)^2]},\ \varepsilon = \sqrt{[(1 + (2\zeta r)^2) / ((1 - r^2)^2 + (2\zeta r)^2)]},\ F_T = \varepsilon F_0$

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2. Force Transmissibility:

- **Q.** Draw the force transmissibility curve and explain the three regions of vibration control. **(WINTER 2023, 4 Marks; SUMMER 2025, 3 Marks)**

Answer:

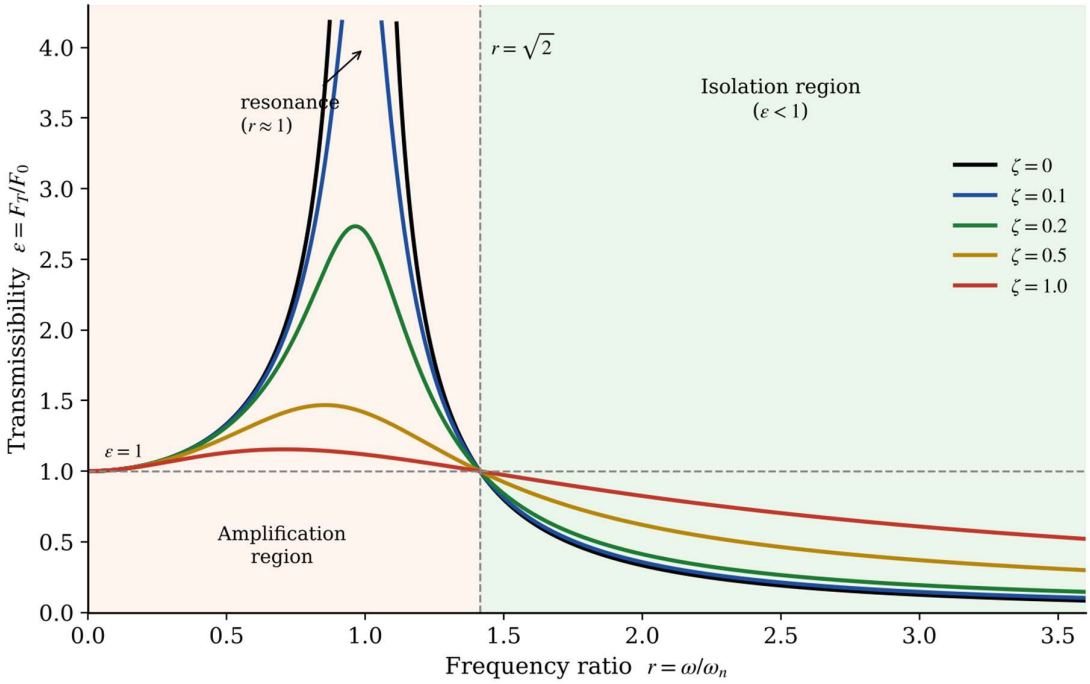


Fig. 18: Force transmissibility versus frequency ratio

Three regions:

- **Region I (0 < r < √2):** Amplification region. Transmissibility ε > 1. Isolation is not effective; damping reduces transmitted force.
- **Region II (r = √2):** All curves intersect at ε = 1. Transmitted force equals exciting force regardless of damping.
- **Region III (r > √2):** Isolation region. Transmissibility ε < 1. Increasing damping increases transmitted force; hence low damping is preferred for effective isolation.

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3. Vibration Isolation:

- **Q.** Explain vibration isolation in detail. / What is vibration isolation? List materials used. **(WINTER 2022, 3 Marks; WINTER 2023, 3 Marks; SUMMER 2025, 3 Marks; SUMMER 2026, 3 Marks OR)**

Answer:

- **Vibration Isolation:** A technique of reducing the transmission of vibration from a machine to its foundation or from a foundation to a system, using isolators (springs, dampers, or elastomers).

Materials used:

- Steel springs (coil or leaf).
- Rubber pads or elastomers.
- Cork and felt.
- Pneumatic isolators.

Condition for effective isolation:

$r = \omega/\omega_n > \sqrt{2} \Rightarrow \omega_n < \omega/\sqrt{2}$

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4. Analysis of a Vibrating System with Damping (Numerical):

- **Q.** A vibrating system is defined by m, k, and c. Determine: (i) Damping factor, (ii) Damped natural frequency, (iii) Logarithmic decrement, (iv) Ratio of two consecutive amplitudes. **(WINTER 2022, 7 Marks; SUMMER 2026, 7 Marks; SUMMER 2026, 7 Marks OR)**

Answer:

Given:

- Mass m = 30 kg
- Spring stiffness k = 20 kN/m = 20000 N/m
- Damping factor ζ = 0.25

To Find:

(a) Critical damping coefficient c_c, (b) Natural frequency of damped vibration f_d, (c) Logarithmic decrement δ, (d) Ratio of two successive amplitudes A_1/A_2.

Formula:

$\omega_n = \sqrt{(k/m)},\ c_c = 2\sqrt{(km)} = 2m\omega_n$
$\omega_d = \omega_n\sqrt{(1-\zeta^2)},\ f_d = \omega_d/2\pi$
$\delta = 2\pi\zeta/\sqrt{(1-\zeta^2)},\ A_1/A_2 = e\text{^}\delta$

Solution:

Natural frequency:

$\omega_n = \sqrt{(20000/30)} = 25.82\ rad/s$

Critical damping coefficient:

$c_c = 2\sqrt{(20000(30))} = 1549.19\ N{\cdot}s/m$

Damped natural frequency:

$\omega_d = 25.82\sqrt{(1-0.25^2)} = 25.0\ rad/s$
$f_d = 25.0/2\pi = 3.98\ Hz$

Logarithmic decrement:

$\delta = 2\pi(0.25)/\sqrt{(1-0.25^2)} = 1.622$

Ratio of successive amplitudes:

$A_1/A_2 = e\text{^}1.622 = 5.06$

Final Answer:

$c_c = 1549.19\ N{\cdot}s/m,\ f_d = 3.98\ Hz,\ \delta = 1.622,\ A_1/A_2 = 5.06$

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5. Equation of Motion for Forced Damped Vibration:

- **Q.** Solve the equation of motion for a forced damped vibration by graphical method. / Illustrate all the forces graphically for forced damped vibration. **(SUMMER 2024, 3 Marks; WINTER 2024, 3 Marks)**

Answer:

Equation of motion:

$m\ddot{x} + c\dot{x} + kx = F_0 \sin\omega t$

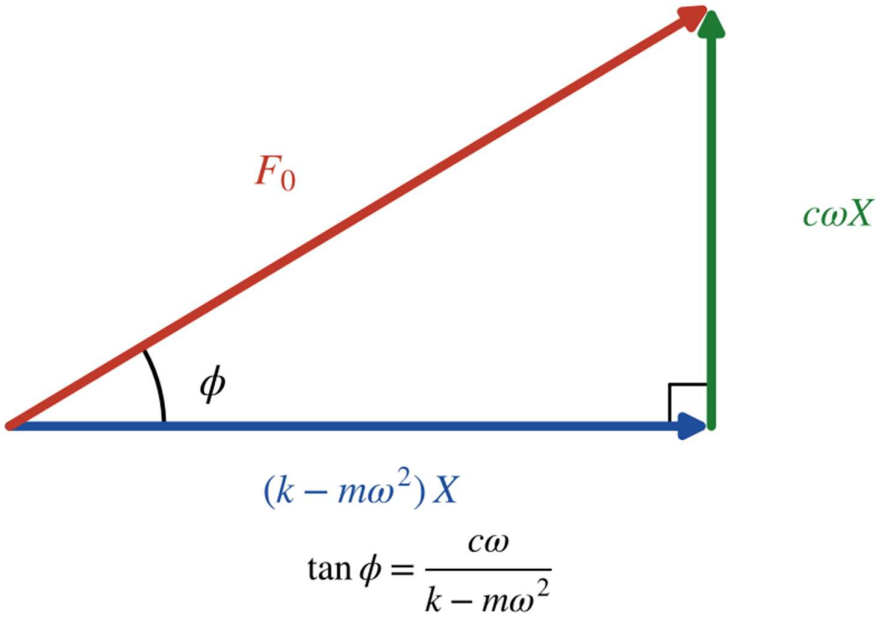
Graphical method (Phasor diagram):

- Inertia force $m\omega^2X$ leads displacement X by 180°.
- Damping force $c\omega X$ leads displacement by 90°.
- Spring force kX is in phase with displacement.
- Resultant of spring and inertia force $(k - m\omega^2)X$ is in phase or out of phase.
- Applied force F_0 is the vector sum of $(k - m\omega^2)X$, $c\omega X$.

From vector triangle:

$F_0 = \sqrt{[((k - m\omega^2)X)^2 + (c\omega X)^2]}$
$X = F_0 / \sqrt{[(k - m\omega^2)^2 + (c\omega)^2]}$

$$F_0 = \sqrt{[(k - m\omega^2)X]^2 + (c\omega X)^2}$$



$$\tan\phi = \frac{c\omega}{k - m\omega^2}$$

Fig. 19: Vector diagram for forced vibration with damping

Other Important Questions:

1. Magnification Factor:

- **Q.** Explain the term Magnification Factor and obtain an expression for it. **(SUMMER 2022, 3 Marks)**

Answer:

- **Magnification Factor:** Ratio of steady-state amplitude to static deflection under constant force F_o.
  $M.F. = X/X_0 = 1 / \sqrt{[(1 - r^2)^2 + (2\zeta r)^2]}$

where $X_o = F_o/k$.

- **Frequency Ratio:** Ratio of excitation frequency to natural frequency.
  $r = \omega/\omega_n$
- At r = 1, resonance occurs; M.F. is maximum. For small damping, maximum M.F. occurs near r = 1. For high r, M.F. tends to zero.

- **Q.** Define Magnification Factor and Frequency Ratio. **(WINTER 2025, 3 Marks OR)**

Answer:

- **Magnification Factor:** Ratio of steady-state amplitude to static deflection under constant force F_o.
  $M.F. = X/X_0 = 1 / \sqrt{[(1 - r^2)^2 + (2\zeta r)^2]}$

where $X_o = F_o/k$.

- **Frequency Ratio:** Ratio of excitation frequency to natural frequency.
  $r = \omega/\omega_n$

- At r = 1, resonance occurs; M.F. is maximum. For small damping, maximum M.F. occurs near r = 1. For high r, M.F. tends to zero.

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2. Forced Vibration of an Engine on a Chassis (Numerical):

- **Q.** A single cylinder vertical petrol engine of total mass 300 kg is mounted upon a steel chassis frame and causes a vertical static deflection of 2 mm. The reciprocating parts of the engine has a mass of 20 kg and move through a vertical stroke of 150 mm with simple harmonic motion. A dashpot is provided whose damping resistance is directly proportional to the velocity and amounts to 1.5 kN per meter per second. Examine the effect of forced vibration considering that the steady state of vibration is reached and determine the amplitude of forced vibrations, when the driving shaft of the engine rotates at 480 r.p.m, and the speed of the driving shaft at which resonance will occur. **(SUMMER 2026, 7 Marks)**

Answer:

Given:

- Total mass M = 300 kg
- Static deflection δ_{st} = 2 mm = 0.002 m
- Reciprocating mass m_{rec} = 20 kg
- Vertical stroke L = 150 mm = 0.15 m
- Damping resistance c = 1.5 kN per m/s = 1500 N·s/m
- Engine speed N = 480 rpm

To Find:

(i) Amplitude of forced vibration X, (ii) Speed at resonance N_r.

Formula:

$$k = Mg/\delta_{st},\ e = L/2$$

$$\omega = 2\pi N/60,\ \omega_n = \sqrt{(k/M)}$$

$$X = m_{rec}\, e\, \omega^2 / \sqrt{[(k - M\omega^2)^2 + (c\omega)^2]}$$

Resonance speed:

$$N_r = 60\omega_n/2\pi$$

Solution:

Stiffness:

$$k = 300(9.81)/0.002 = 1{,}471{,}500\ \text{N/m}$$

Eccentricity of reciprocating mass:

$$e = 0.15/2 = 0.075\ \text{m}$$

Natural frequency:

$$\omega_n = \sqrt{(1471500/300)} = 70.04\ \text{rad/s}$$

Resonance speed:

$$N_r = 60(70.04)/2\pi = 668.8\ \text{rpm}$$

Engine excitation frequency at 480 rpm:

$$\omega = 2\pi(480)/60 = 50.27\ \text{rad/s}$$

Excitation force amplitude:

$$F_0 = m_{rec}\, e\, \omega^2 = 20(0.075)(50.27)^2 = 3790.6\ \text{N}$$

Denominator:

$$k - M\omega^2 = 1471500 - 300(2527) = 713400$$

$$c\omega = 1500(50.27) = 75405$$

$$\sqrt{(713400^2 + 75405^2)} = 717376$$

Amplitude:

$X = 3790.6/717376 = 0.00528\ m = 5.28\ mm$

Final Answer:

$X = 5.28\ mm,\ N_r = 668.8\ rpm$

---

3. Effect of Damping on Forced Vibration:

- **Q.** Analyze the effect of damping on frequency forced vibrations under under-damped condition using graphical method. **(SUMMER 2026, 7 Marks OR)**

Answer:

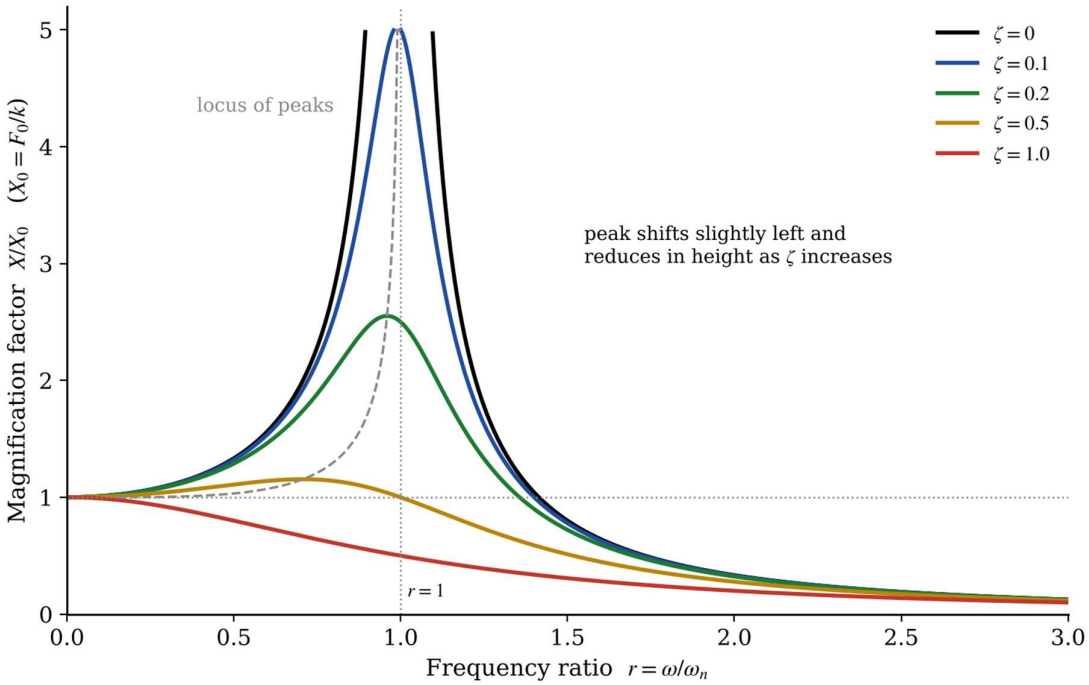


Fig. 20: Effect of damping on forced vibration (magnification factor)

Explanation:

- **At r = 0:** All curves give M.F. = 1, independent of damping.
- **Near resonance (r ≈ 1):** Amplitude is maximum. For ζ = 0, it is infinite; for ζ > 0, finite peak.
- **Peak amplitude condition:** Differentiate M.F. w.r.t. r and equate to zero:
  $r_peak = \sqrt{(1 - 2\zeta^2)}$
- **Peak M.F.:**
  $M.F._max = 1 / (2\zeta\sqrt{(1 - \zeta^2)})$
- **For high frequencies (r ≫ 1):** M.F. approaches zero; damping increases the amplitude compared to undamped case.
- **Phase angle:**
  $\varphi = \tan^{-1}(2\zeta r / (1 - r^2))$

At resonance, ϕ = 90° irrespective of damping.

- **Isolation region:** For r > √2, increased damping increases transmitted force, hence low damping is desirable.

Chapter 7: Critical / Whirling Speed and Shaft Vibrations

Repeated Questions:

1. Critical / Whirling Speed of a Shaft:

- **Q.** Explain the whirling speed of a shaft. / Define critical or whirling speed of a shaft. **(WINTER 2023, 3 Marks; SUMMER 2024, 3 Marks)**

Answer:

- **Critical (Whirling) Speed:** The speed of a rotating shaft at which the shaft tends to vibrate violently in the transverse direction due to resonance.
- Occurs when the rotational frequency of the shaft equals its natural frequency of transverse vibration.
- At this speed, the centrifugal force due to a small initial eccentricity causes excessive deflection.

Formula for simply supported shaft with a single rotor of mass m and stiffness k:

$$\omega_c = \sqrt{(k/m)} = \sqrt{(g/\delta)}$$

where δ is the static deflection.

In rpm:

$$N_c = (60/2\pi)\sqrt{(g/\delta)} \approx 29.9 / \sqrt{\delta} \quad (\delta \text{ in m})$$

- **Q.** Derive the formula for the critical speed of a shaft carrying a single rotor without damping. **(WINTER 2023, 4 Marks; SUMMER 2025, 4 Marks)**

Answer:

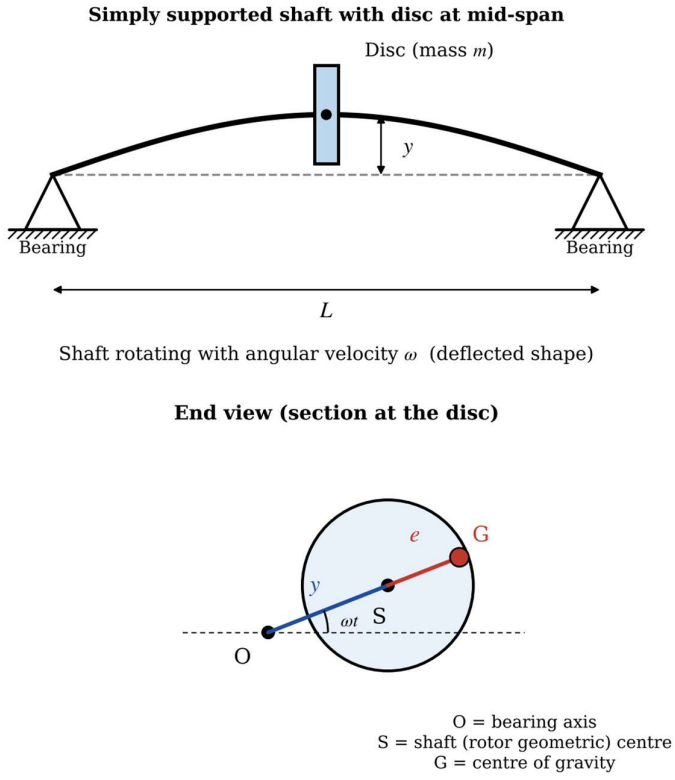


Fig. 21: Whirling of a shaft carrying a single rotor

Derivation:

1. Let y be the transverse deflection of the shaft at the disc. The total eccentricity of the CG from the shaft axis is (y + e).
2. The centrifugal force acting outward is:

$$F_c = m\omega^2(y + e)$$

3. The restoring elastic force of the shaft acting inward is:

$$F_r = ky$$

where k is the stiffness of the shaft.

4. Equating the forces for dynamic equilibrium:

$$ky = m\omega^2(y + e)$$

5. Rearranging for deflection y:

$$y(k - m\omega^2) = me\omega^2$$

$$y = me\omega^2 / (k - m\omega^2) = e / [(k/m\omega^2) - 1]$$

6. The deflection y tends to infinity when the denominator approaches zero:

$$k - m\omega^2 = 0$$

$$\omega_c = \sqrt{(k/m)}$$

Final Result:

The critical speed is:

$$\omega_c = \sqrt{(k/m)}$$

---

2. Critical Speed using Dunkerley's Method (Numerical):

- **Q.** A shaft of length L and diameter D, simply supported, carries multiple rotors. Find the critical speed of the shaft using Dunkerley's method. **(SUMMER 2024, 7 Marks; SUMMER 2025, 7 Marks)**

Answer:

Given:

- Simply supported shaft of length L = 250 mm = 0.25 m (from Figure)
- Diameter D (implied standard, or given as $I = 40 \times 10^4$ mm^4 = 4×10^{-7} m^4)
- Mass 1: m_1 = 60 kg at a_1 = 50 mm = 0.05 m from left bearing
- Mass 2: m_2 = 100 kg at a_2 = 150 mm = 0.15 m from left bearing
- Modulus of elasticity for steel, E = 200 GPa = 200×10^9 N/m²
- Neglect damping.
- **To Find:** Fundamental frequency of vibration f_n (or N_c).

Formula:

Static deflection at any point due to a point load at distance a from left support and b from right support (where L = a + b):

$$\delta = W a^2 b^2 / (3EIL)$$

where W = mg.

Dunkerley's equation:

$$1/f_n^2 = 1/f_1^2 + 1/f_2^2$$

$$f_1 = (1/2\pi)\sqrt{(g/\delta_{11})},\ f_2 = (1/2\pi)\sqrt{(g/\delta_{22})}$$

Solution:

1. Deflection at 1 due to load at 1 (δ_{11}): a_1 = 0.05 m, b_1 = 0.2 m

$$W_1 = 60 \times 9.81 = 588.6 \text{ N}$$

$$\delta_{11} = 588.6 \times (0.05)^2 \times (0.2)^2 / (3 \times (200\times10^9) \times (4\times10^{-7}) \times 0.25)$$

Denominator = $3 \times 200\times10^9 \times 4\times10^{-7} \times 0.25 = 60000$

Numerator = 588.6 × 0.0025 × 0.04 = 0.05886

$\delta_{11} = 0.05886/60000 = 9.81 \times 10^{-7}$ m

2. Frequency f_1:

$f_1 = (1/2\pi)\sqrt{(9.81/(9.81\times10^{-7}))} = (1/2\pi)(3162.3) = 503.3$ Hz

3. Deflection at 2 due to load at 2 (δ_{22}): $a_2 = 0.15$ m, $b_2 = 0.1$ m

$W_2 = 100 \times 9.81 = 981$ N

$\delta_{22} = 981 \times (0.15)^2 \times (0.1)^2 / 60000 = 0.2207/60000 = 3.679 \times 10^{-6}$ m

4. Frequency f_2:

$f_2 = (1/2\pi)\sqrt{(9.81/(3.679\times10^{-6}))} = (1/2\pi)(1633) = 259.9$ Hz

5. Dunkerley's Equation:

$1/f_n^2 = 1/(503.3)^2 + 1/(259.9)^2$

$1/f_n^2 = 3.95 \times 10^{-6} + 1.48 \times 10^{-5} = 1.875 \times 10^{-5}$

$f_n^2 = 53320 \Rightarrow f_n = 230.9$ Hz

Final Answer:

$f_n = 230.9$ Hz or $N_c = 13856$ rpm

---

3. Shaft with Single Rotor (Numerical):

- **Q.** A rotor of mass M is mounted midway on a shaft of diameter D and length L. The CG is offset by e. If the rotor rotates at N rpm, find the amplitude of steady-state vibrations and the dynamic force transmitted to the bearings. Neglect damping. **(WINTER 2023, 7 Marks; WINTER 2024, 7 Marks)**

Answer:

Given:

Mass M, shaft diameter D, length L, eccentricity e, speed N rpm. (Assume simply supported at ends, disc at mid-span).

$k = 48EI/L^3,\ \omega = 2\pi N/60,\ \omega_n = \sqrt{(k/M)}$

- **To Find:** Amplitude of steady-state vibration y, dynamic force transmitted F_d.

Formula:

$y = e / [(\omega_n/\omega)^2 - 1]$

$F_d = ky$

Solution:

1. Calculate moment of inertia:

$I = \pi D^4/64$

2. Calculate stiffness:

$k = 48EI/L^3$

3. Calculate natural frequency:

$\omega_n = \sqrt{(k/M)}$

4. Calculate rotational speed ω.

5. Substitute in amplitude formula:

$y = e / [(\omega_n/\omega)^2 - 1]$

6. Calculate dynamic force:

$F_d = M\omega^2(y + e)$

(or F_d = ky)

Final Answer:

$y = e / [(\omega_n/\omega)^2 - 1],\ F_d = ky$

Other Important Questions:

1. Critical Speed considering Shaft Mass (Numerical):

- **Q.** A shaft of length L, supported at ends, carries two wheels. The shaft is hollow (given dimensions). Taking into account the mass of the shaft, find the lowest whirling speed. **(SUMMER 2024, 7 Marks OR)**

Answer:
[Answer not found in the answer chat – to be added]

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2. Whirling of Shafts (Theory):

- **Q.** Explain the term 'Half frequency whirl'. Derive the governing equation for the same. **(WINTER 2025, 7 Marks)**

Answer:
Half-frequency whirl is a self-excited vibration phenomenon in fluid-film journal bearings. The shaft whirls at approximately half the rotational speed (ω_whirl ≈ 0.5ω).

Derivation:

1. Consider a journal bearing with oil film. The journal has an eccentricity e from the bearing centre.
2. The oil film pressure creates a resultant force F that can be resolved into radial and tangential components.
3. The tangential force component drives the whirl motion, acting like a negative damping force.
4. The equations of motion for the journal are:

$$m\ddot{x} + c\dot{x} + kx = F_{oil,x}$$
$$m\ddot{y} + c\dot{y} + ky = F_{oil,y}$$

5. For small perturbations, the oil force is proportional to the displacement and velocity. The instability occurs when the cross-coupled stiffness forces exceed the direct damping forces.
6. The whirl frequency is found to be:

$$\omega_{whirl} = \omega/2$$

and the whirl ratio (whirl frequency / rotational frequency) is approximately 0.5.

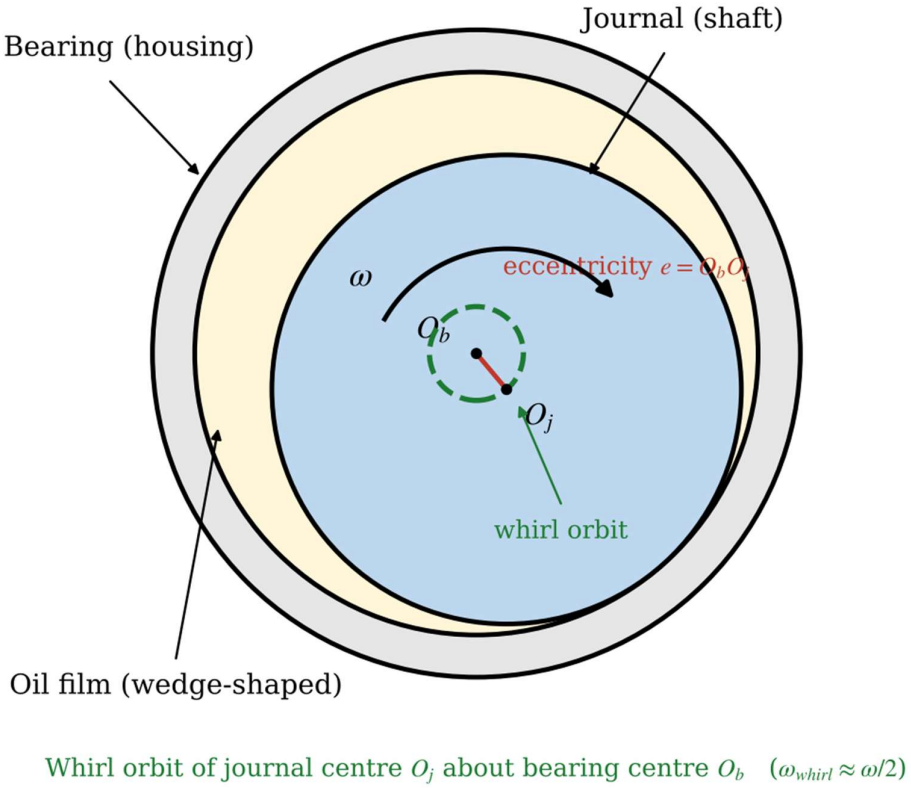


Fig. 22: Journal bearing: whirl orbit and oil film (half-frequency whirl)

- **Q.** With usual notations; Derive the equation to find the natural frequency of shaft carrying several loads undergoes transverse vibrations using Dunkerley's method. **(WINTER 2025, 7 Marks OR)**

Answer:

- **Dunkerley's Empirical Formula:**

$$1/f_c^2 = 1/f_1^2 + 1/f_2^2 + \cdots + 1/f_n^2$$

where f_c is the fundamental (lowest) critical frequency, and f_1, f_2, …, f_n are the natural frequencies of the shaft carrying each load individually.

Derivation/Explanation (Rayleigh's Method):

1. Assume the shaft vibrates in its fundamental mode with deflection y(x).
2. The maximum kinetic energy is:

$$T_{max} = \tfrac{1}{2}\omega^2 \int m(x)\, y^2\, dx + \tfrac{1}{2}\omega^2 \Sigma M_i\, y_i^2$$

3. The maximum potential energy is:

$$U_{max} = \tfrac{1}{2} \int EI\, (d^2y/dx^2)^2\, dx$$

4. For a system with point loads, Rayleigh's formula gives:

$$\omega_n^2 = \Sigma W_i y_i \,/\, (\Sigma W_i y_i^2 / g)$$

5. Dunkerley modified this for multiple loads to find the lower bound of critical speed:

$$1/\omega_c^2 = 1/\omega_1^2 + 1/\omega_2^2 + \cdots + 1/\omega_n^2$$

where $\omega_i^2 = g/\delta_{ii}$, and δ_{ii} is the static deflection at load i due to load i acting alone.

Final Equation in rpm:

$$1/N_c^2 = 1/N_1^2 + 1/N_2^2 + \cdots + 1/N_n^2$$

- **Q.** Prove that the critical or whirling speed is the same as the natural frequency of transverse vibration with usual notations. **(WINTER 2025, 4 Marks)**

Answer:

The natural frequency of transverse vibration of a mass m on a shaft of stiffness k is:

$$\omega_n = \sqrt{(k/m)}$$

From the derivation of critical speed, the speed at which the shaft deflects infinitely is:

$$\omega_c = \sqrt{(k/m)}$$

Therefore, equating the two:

$$\omega_c = \omega_n$$

- **Physical significance:** Whirling occurs due to resonance; when the excitation frequency (rotational speed) matches the system's natural frequency of transverse vibration.

In terms of static deflection $\delta = mg/k = g/\omega_n^2$, the natural frequency is:

$$f_n = (1/2\pi)\sqrt{(g/\delta)}$$

which is the same as the critical speed in Hz.

---

3. Critical Speed – Resonance, Damping and Safety:

- **Q.** Propose design modifications to prevent resonance in high-speed rotors. **(SUMMER 2026, 3 Marks)**

Answer:

- **Change natural frequency:** Alter mass or stiffness (e.g., change shaft diameter, bearing span) so that ω_n is far from operating speed.
- **Add damping:** Use squeeze-film dampers, elastomeric supports, or friction dampers to reduce amplitude at resonance.
- **Vibration isolation:** Mount rotor on isolators to reduce transmitted forces.
- **Avoid operating near ω_n:** Design the operating speed to be either well below or well above (usually > $\sqrt{2}\ \omega_n$) the critical speed.

- **Q.** Propose safety considerations for operating near critical speed. **(SUMMER 2026, 3 Marks OR)**

Answer:

- **Quick passage:** Accelerate rapidly through the critical speed; do not dwell at resonance.
- **Robust bearings:** Use high-capacity bearings to withstand dynamic loads during startup/shutdown.
- **Continuous monitoring:** Install vibration sensors to detect excessive amplitudes and trigger shutdown.
- **Proper lubrication:** Ensure oil film stability to prevent half-frequency whirl.

- **Q.** Analyze the effect of damping on the critical speed of a horizontal shaft. **(SUMMER 2026, 4 Marks)**

Answer:

- **With damping, the amplitude of vibration at resonance becomes finite instead of infinite:**

$$y_{max} = e/(2\zeta)$$

- **The peak amplitude occurs at a frequency slightly lower than the undamped critical speed:**

$$\omega_{peak} = \omega_n \sqrt{(1 - 2\zeta^2)}$$

- The critical speed (resonance) is still approximately equal to the undamped natural frequency for small damping.

- Damping increases the safe operating range and reduces dynamic forces transmitted to bearings.

- **Q.** Examine the effect of resonance in rotating shafts. **(SUMMER 2026, 4 Marks OR)**

Answer:

- **Large amplitudes:** At resonance, even a small eccentricity causes large deflections, potentially exceeding clearance limits.
- **High dynamic forces:** Forces on bearings increase drastically, causing fatigue and wear.
- **Noise and vibration:** Excessive vibration is transmitted to the foundation, causing discomfort and structural damage.
- **Catastrophic failure:** If not controlled, resonance can lead to shaft fracture, bearing seizure, or complete machine failure.

---

4. Fundamental Frequency of a Shaft with Two Discs (Numerical):

- **Q.** A solid shaft of uniform diameter is supported by two short bearings as shown in Figure 1. The above shaft carries two discs having concentrated masses of 60 kg and 100 kg respectively. Determine the fundamental frequency of vibration if the moment of inertia of the shaft is 40×10^4 mm^4 and damping at the bearing is neglected. [Diagram required – refer to the original question paper] **(SUMMER 2026, 7 Marks)**

Answer:

Given:

- Simply supported shaft of length L = 250 mm = 0.25 m (from Figure)
- Diameter D (implied standard, or given as $I = 40 \times 10^4$ $mm^4 = 4 \times 10^{-7}$ m^4)
- Mass 1: $m_1 = 60$ kg at $a_1 = 50$ mm = 0.05 m from left bearing
- Mass 2: $m_2 = 100$ kg at $a_2 = 150$ mm = 0.15 m from left bearing
- Modulus of elasticity for steel, $E = 200$ GPa $= 200 \times 10^9$ N/m^2
- Neglect damping.
- **To Find:** Fundamental frequency of vibration f_n (or N_c).

Formula:

Static deflection at any point due to a point load at distance a from left support and b from right support (where L = a + b):

$$\delta = W a^2 b^2 / (3EIL)$$

where W = mg.

Dunkerley's equation:

$$1/f_n^2 = 1/f_1^2 + 1/f_2^2$$

$$f_1 = (1/2\pi)\sqrt{(g/\delta_{11})},\ f_2 = (1/2\pi)\sqrt{(g/\delta_{22})}$$

Solution:

1. Deflection at 1 due to load at 1 (δ_{11}): $a_1 = 0.05$ m, $b_1 = 0.2$ m

$$W_1 = 60 \times 9.81 = 588.6 \text{ N}$$

$$\delta_{11} = 588.6 \times (0.05)^2 \times (0.2)^2 / (3 \times (200\times10^9) \times (4\times10^{-7}) \times 0.25)$$

Denominator = $3 \times 200\times10^9 \times 4\times10^{-7} \times 0.25 = 60000$

Numerator = $588.6 \times 0.0025 \times 0.04 = 0.05886$

$$\delta_{11} = 0.05886/60000 = 9.81 \times 10^{-7} \text{ m}$$

2. Frequency f_1:

$$f_1 = (1/2\pi)\sqrt{(9.81/(9.81\times10^{-7}))} = (1/2\pi)(3162.3) = 503.3 \text{ Hz}$$

3. Deflection at 2 due to load at 2 (δ_{22}): $a_2 = 0.15$ m, $b_2 = 0.1$ m

$$W_2 = 100 \times 9.81 = 981 \text{ N}$$

$\delta_{22} = 981 \times (0.15)^2 \times (0.1)^2 / 60000 = 0.2207/60000 = 3.679 \times 10^{-6}$ m

4. Frequency f_2:

$f_2 = (1/2\pi)\sqrt{(9.81/(3.679\times10^{-6}))} = (1/2\pi)(1633) = 259.9$ Hz

5. Dunkerley's Equation:

$1/f_n^2 = 1/(503.3)^2 + 1/(259.9)^2$

$1/f_n^2 = 3.95 \times 10^{-6} + 1.48 \times 10^{-5} = 1.875 \times 10^{-5}$

$f_n^2 = 53320 \Rightarrow f_n = 230.9$ Hz

Final Answer:

f_n = 230.9 Hz or N_c = 13856 rpm

Chapter 8: Uncategorised Questions

Repeated Questions:

None identified.

Other Important Questions:

1. Statics (Lami's Theorem):

- **Q.** State Lami's theorem and give suitable example. **(WINTER 2025, 3 Marks)**

Answer:

- **Lami's Theorem:** If three coplanar, concurrent, and non-collinear forces are in equilibrium, then each force is proportional to the sine of the angle between the other two forces.

  $P/\sin\alpha = Q/\sin\beta = R/\sin\gamma$

where α, β, and γ are the angles opposite to forces P, Q, and R respectively.

- **Suitable Example:** A sphere of weight W resting in a V-shaped trough, or a particle suspended by two strings. The three forces are the weight acting downward and the two tension forces in the strings.

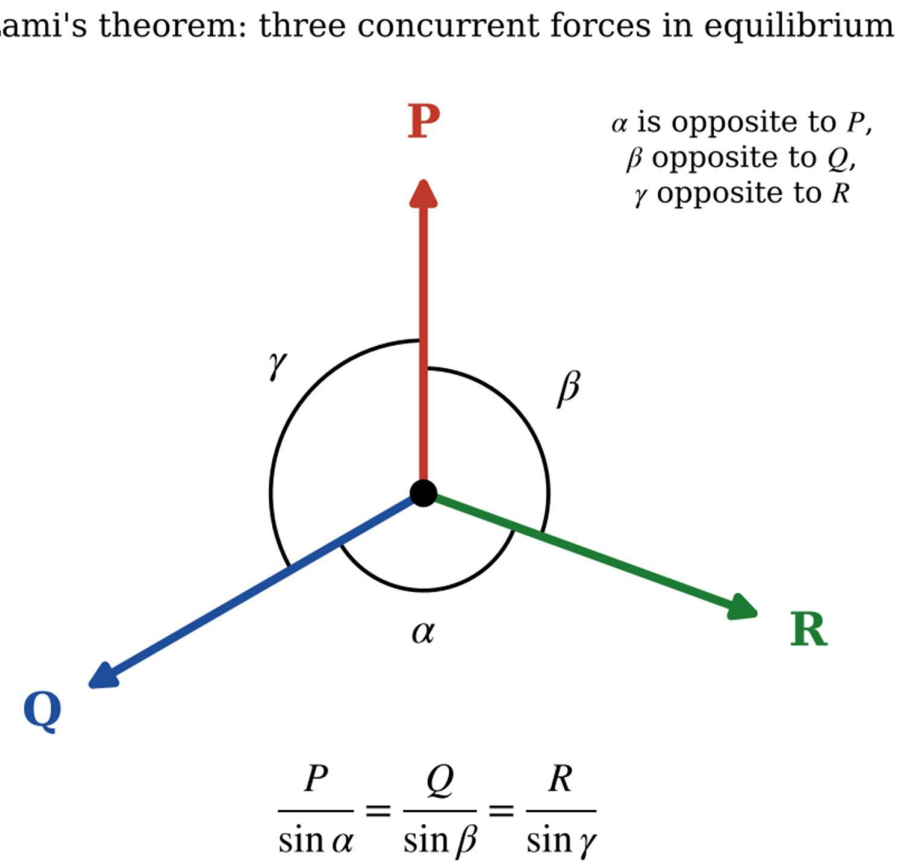


Fig. 23: Three concurrent forces in equilibrium (Lami's theorem)

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